# SYMPLECTIC 3-ALGEBRAS AND $D = 3$, $\mathcal{N} = 4, 5, 6, 8$ SUPERCONFORMAL CHERN-SIMONS-MATTER THEORIES

by

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# ABSTRACT

M-theory is the underlying theory of five different string theories and 11D supergravity theory. While strings (1+1D) are fundamental objects in string theory, M2-branes (1+2D) are fundamental objects in M-theory. According to the Gauge/Gravity duality, a gravity theory is equivalent to a gauge theory. Extended ($\mathcal{N} \geq 4$) superconformal Chern-Simons-matter (CSM) theories in 3D are natural candidates of the dual gauge theories of multi M2-branes.

In the last two years, the $\mathcal{N} = 4, 5, 6$ CSM theories were constructed by using ordinary Lie 2-algebras, and the $\mathcal{N} = 8$ theory was constructed by using 3-algebra. However, it remains unclear whether these theories can be constructed in a unified 3-algebra approach or not. It is also natural to ask whether there are new examples of the extended superconformal CSM theories.

In this thesis, we propose to solve these two problems. We define a 3-algebra with structure constants being *symmetric* in the first two indices. We also introduce an invariant antisymmetric tensor into this 3-algebra and call it a *symplectic* 3-algebra. The $D = 3, \mathcal{N} = 4, 5, 6, 8$ CSM theories are constructed in terms of this unified 3-algebraic structure, and some new examples of the $\mathcal{N} = 4$ quiver gauge theories are derived as well. In particular, in order to realize the 3-algebra used to construct the $\mathcal{N} = 4$ quiver gauge theories, we 'fuse' two simple super Lie algebras into a single new super Lie algebra, by requiring that the even parts of these two simple super Lie algebras share one simple factor. We demonstrate how to construct this class of new super Lie algebras by presenting an explicit example. Finally, a quantization scheme for the 3-brackets is proposed.

To Jing Bian.

## CONTENTS

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# CHAPTER 1

# INTRODUCTION

String theory is a plausible candidate for unifying quantum gravity and elementary particle forces. There are five different string theories. All known string theories and 11D supergravity theory arise as different limits of a single theory: M-theory. M2-branes (1+2D) are important in that they are fundamental objects in M-theory. According to the Gauge/Gravity duality in string theory, a non-Abelian gauge theory is equivalent to a quantum gravity theory. In the last two years, extended ($\mathcal{N} \geq 4$)[1] supersymmetric Chern-Simons-matter (CSM) theories in 3D have attracted a lot of interest in the string/M theory community, because they are natural candidates of the dual gauge theories of multi M2-branes in M-theory. For example, in the next page we will see that M-theory on $AdS_4 \times S^7/\mathbf{Z}_k$ ($k > 2$) is equivalent to an $\mathcal{N} = 6$ superconformal CSM theory in 3D. These two theories have the same amount of supersymmetries.

Less extended supersymmetric ($\mathcal{N} < 4$) CSM theories with arbitrary gauge groups were constructed and investigated long time ago [1]-[6]. (To our knowledge, their dual gravity theories are still under construction.) And generic Chern-Simons gauge theories with or without (massless) matter were demonstrated to be conformally invariant even at the quantum level [7, 8, 9, 10, 11]. However, it was much more difficult until recently to construct $\mathcal{N} \geq 4$ CSM theories, since only some special gauge groups are allowed in these theories.

By virtue of the Nambu 3-algebra structure [12, 13], the maximally supersymmetric $\mathcal{N} = 8$ CSM theory with $SO(4)$ gauge group was first constructed independently by Bagger and Lambert [14] and by Gustavsson [15] (BLG). The BLG theory was conjectured to be the dual gauge theory of two M2-branes [16, 17, 18, 19]. The Nambu 3-algebra, equipped with a symmetric and positive-definite metric, has the limitation that it can only

[1] Here '$\mathcal{N}$' stands for $\mathcal{N}$ copies of supersymmetries. In 3D, if $\mathcal{N} = 1$, there are two independent fermionic generators.

generate an $SO(4)$ gauge theory [20, 21], much too restrictive for a low-energy effective description of M2-branes.

Very soon Aharony, Bergman, Jafferis and Maldacena (ABJM) observed [25] that an $\mathcal{N} = 2$ superconformal CSM theory, with gauge group $U(N) \times U(N)$, actually has an $SU(4)$ R-symmetry, hence an enhanced supersymmetry $\mathcal{N} = 6$. The same theory was also obtained by taking the infrared limit of a brane construction. In their formulation, the Nambu 3-algebra structure did not play any role, though the ABJM theory with $SU(2) \times SU(2)$ gauge group is equivalent to the BLG theory. Based on the brane construction, ABJM conjectured that at level $k$ their theory describes the low energy limit of $N$ M2-branes probing a $\mathbf{C}^4/\mathbf{Z}_k$ singularity. In the special cases of $k = 1, 2$, the theory has the maximal supersymmetries ($\mathcal{N} = 8$) [25, 26, 27, 28]. In a large-$N$ limit the ABJM theory is then dual to M-theory on $AdS_4 \times S^7/\mathbf{Z}_k$ [25]. The superspace formulation and a manifest $SU(4)$ R-symmetry formulation of the ABJM theory can be found in Ref. [29] and [30], respectively.

In Ref. [31, 32], some extended superconformal gauge theories were constructed by taking a conformal limit of $D = 3$ gauged supergravity theories. In this approach, the embedding tensors play a crucial role. Gaiotto and Witten (GW) [33] have been able to construct a large class of $\mathcal{N} = 4$ CSM theories by a method that enhances $\mathcal{N} = 1$ supersymmetry to $\mathcal{N} = 4$. They also demonstrated that the gauge groups can be classified by super Lie algebras. In Ref. [34], the GW theory was extended to include additional twisted hypermultiplets; in particular, the extended GW theory with $SO(4)$ gauge group was demonstrated to be equivalent to the BLG theory. In Ref. [35], two new theories, $\mathcal{N} = 5$, $Sp(2M) \times O(N)$ and $\mathcal{N} = 6$, $Sp(2M) \times O(2)$ CSM theories, were constructed by further enhancing the R-symmetry to $Sp(4)$ and $SU(4)$, respectively, and the $\mathcal{N} = 6$, $U(M) \times U(N)$ CSM theory was rederived. The gravity duals of $\mathcal{N} = 5$, $Sp(2M) \times O(N)$ and $\mathcal{N} = 6$, $U(M) \times U(N)$ theories were studied in Ref. [36]. By using group representation theory and applying GW's super-Lie-algebra method for classifying gauge groups, the $\mathcal{N} = 1$ to $\mathcal{N} = 8$ CSM theories were constructed systematically in a recent paper [37].

The progress mentioned in the last two paragraphs was made using mainly ordinary Lie algebras. On the other hand, Bagger and Lambert (BL) have been able to construct the $\mathcal{N} = 6, U(M) \times U(N)$ theory in terms of a modified 3-algebra [38]. Unlike the Nambu 3-algebra with totally antisymmetric structure constants, the structure constants of the modified 3-algebra are antisymmetric only in the first two indices. By introducing an invariant antisymmetric tensor into a 3-algebra, hence called a 'symplectic 3-algebra', another class of $\mathcal{N} = 6$ CSM theories, with gauge group $Sp(2M) \times O(2)$, has been constructed in Ref. [39]. It is also demonstrated that the $\mathcal{N} = 6, U(M) \times U(N)$ theory can be recast into the symplectic 3-algebraic formalism [39]. In Ref. [40], both the general $\mathcal{N} = 5$ and $\mathcal{N} = 6$ CSM theories have been formulated in a unified symplectic 3-algebraic framework. These theories based on 3-algebras are constructed by requiring that the supersymmetries be closed on-shell.

The main goal of the thesis is to combine the superspace formalism with the symplectic 3-algebra, then construct all $D = 3$ extended ($\mathcal{N} = 4, 5, 6, 8$) superconformal CSM theories in a unified symplectic 3-algebraic framework. The ordinary Lie algebra counterparts of these superconformal CSM theories are derived by using a super Lie algebra to realize the symplectic 3-algebra. We also derive all known examples of the $D = 3$, $\mathcal{N} = 4, 5, 6, 8$ superconformal CSM theories, and construct some new example of the $\mathcal{N} = 4$ quiver gauge theories.[2]

We first combine the superspace formalism with the 3-algebra, then rederive the general $\mathcal{N} = 5$ theories by using the Giatto-Witten enhancement mechanism. Previously the $\mathcal{N} = 5$ theories were derived from the $\mathcal{N} = 4$ theories by carefully choosing the gauge groups [35, 37]. So the construction of $\mathcal{N} = 5$ theories by enhancing $\mathcal{N} = 1$ supersymmetry is interesting in its own right, especially in a 3-algebraic framework. It provides insight into the relationship between the 3-algebra and conventional Lie-algebra approach.

We then construct general $\mathcal{N} = 4$ theories in the (quaternion) 3-algebra framework, in which there are two similar sets of complex 3-algebra generators. These $\mathcal{N} = 4$ theories are 3-algebra version of Chern-Simons quiver gauge theories. We start from $\mathcal{N} = 5$ super-multiplets, decompose them and the symplectic 3-algebra generators properly, and propose a new superpotential which is $\mathcal{N}$=4 superconformally invariant.

We demonstrate that the $\mathcal{N} = 5$ supersymmetry can be enhanced to $\mathcal{N} = 6$ by decomposing the symplectic 3-algebra and the fields properly, and the fundamental identity and the symmetry and reality properties of the structure constants of the hermitian 3-algebra (used to construct $\mathcal{N} = 6$ theories) can be derived from their $\mathcal{N} = 5$ counterparts.

[2]The gauge group of a quiver gauge theory is a product of $g_i$ factors; and the matter fields are in the bifundamental representations.

In the special case that the structure constants are totally antisymmetric, the hermitian algebra becomes the Nambu 3-algebra. As a result, the $\mathcal{N}=6$ supersymmetry is promoted to $\mathcal{N}=8$, and the corresponding theory becomes the BLG theory.

Therefore $\mathcal{N}=4,5,6,8$ superconformal CSM theories are described by a unified (sympletic) 3-algebraic framework.

We systematically investigate the relations between the 3-algebras, Lie superalgebras, ordinary Lie algebras and embedding tensors that are used to build $D=3$ extended supergravity theories in Ref. [32]. The relations between the 3-algebras and Lie superalgebras are explored in Ref. [37, 42, 46], using representation theory. They did not discuss the relations between the embedding tensors in Ref. [32] and 3-algebras or Lie superalgebras. We fill this gap by a more physical approach.

We demonstrate that the symplectic 3-algebra can be realized in terms of a super Lie algebra. The generators of the 3-algebra $T_I$ can be realized as the fermionic generators of the super Lie algebra $Q_I$, and the 3-bracket is realized in terms of a double graded bracket: $[T_I, T_J; T_K] \doteq [\{Q_I, Q_J\}, Q_K]$. In this realization, the fundamental identity (FI) of the symplectic 3-algebra can be converted into the $MMQ$ Jacobi identity of the super Lie algebra ($M$ is a bosonic generator). It will be shown that the structure constants of the symplectic 3-algebra furnish a quaternion representation of the bosonic part of the super Lie algebra, and play the role of Killing-Cartan metric of the bosonic part of the super Lie algebra. Then the FI of the 3-algebra are rewritten as ordinary commutator, whose structure constants are totally antisymmetric. Moreover, we prove that the structure constants of the symplectic 3-algebra are the components of the embedding tensor proposed in [32], if we realize the symplectic 3-algebra in terms of the super Lie algebra.

The general $\mathcal{N}=5,6,8$ superconformal Chern-Simons-matter theories in terms of ordinary Lie algebras can be rederived from our super-Lie-algebra realization of the symplectic 3-algebras. Not only all known examples of $\mathcal{N}=4,5,6,8$ ordinary CSM theories, but also $\mathcal{N}=4$ CSM quiver gauge theories (including some new examples), can be produced as well. Therefore, our superspace formulation for the super-Lie-algebra realization of symplectic 3-algebras provide a unified treatment of all known $\mathcal{N}=4,5,6,8$ CSM theories, including new examples of $\mathcal{N}=4$ quiver gauge theories as well.

In order to classify the gauge groups of the $\mathcal{N}=4$ quiver gauge theories, we 'fuse' two simple super Lie algebras into a single super Lie algebra, by requiring that the bosonic parts of these two simple super Lie algebras share one simple factor. As a result, the fermionic generators $Q_a$ and $Q_{b'}$ of this pair of super Lie algebras have nontrivial anticommutators, i.e., $\{Q_a, Q_{b'}\} \neq 0$. An explicit example is presented to demonstrate how to construct this kind of new super Lie algebras: we 'fuse' the simple super Lie algebras $OSp(N_2|2N_1)$ and $OSp(N_2|2N_3)$ $(N_1 \neq N_3)$ into a single super Lie algebra, whose bosonic part is the Lie algebra of the group $Sp(2N_1) \times SO(2N_2) \times Sp(2N_3)$ $(N_1 \neq N_3)$. This group can be selected as a gauge group of the $\mathcal{N} = 4$ quiver gauge theory.

The ($\mathcal{N} = 6$) hermitian 3-algebras and the Nambu 3-algebras are also realized in terms of super Lie algebras. We propose a quantization scheme for the symplectic, hermitian and Nambu 3-brackets, by promoting the generators and the double graded commutators of the corresponding super Lie algebras as quantum mechanical operators and double graded commutators, respectively.

We also derive the same $\mathcal{N} = 4, 5, 6$ theories by requiring that the supersymmetry transformations are closed on-shell, i.e., we also examine the closure of the $\mathcal{N} = 4, 5, 6$ algebras. The closure of $\mathcal{N} = 4$ algebra in the GW theory (*without* the twisted hypermultiplets) has been checked in Ref. [33]. However, to our knowledge, the closure of the algebra in theories *with* the twisted hypermultiplets has not been explicitly checked in the literature. So our calculation will fill this gap. The closure of $\mathcal{N} = 6$ algebra is first checked in Ref. [38], using a hermitian 3-algebra approach. Our approach to the $\mathcal{N} = 6$ theories is slightly different from that of [38], in that we use the symplectic 3-algebra to construct the $\mathcal{N} = 6$ theories. As a result, ours is more suited to the case with gauge group $Sp(2N) \times U(1)$.

This thesis is organized as follows. In Chapter 2, we introduce the symplectic three-algebras and define the notations. Section 3.1 is devoted to the construction of the $\mathcal{N} = 5$ theories by enhancing the supersymmetry from $\mathcal{N} = 1$ to $\mathcal{N} = 5$ in a 3-algebraic framework. The (on-shell) closure of the $\mathcal{N} = 5$ algebra is explicitly verified in section 3.2. In section 4.1, we derive the $\mathcal{N}$=4 theories by decomposing the $\mathcal{N} = 5$ super-multiplets and the symplectic 3-algebra properly and proposing a new superpotential. The closure of the $\mathcal{N} = 4$ super algebra is explicitly verified in section 4.2. In Chapter 5, we discuss the relations between 3-algebras, super Lie algebras, ordinary Lie algebras and the embedding tensors proposed in Ref. [32]. In section 6.1, we present how to reproduce the general Lie algebra version of $\mathcal{N} = 5$ theory from the 3-algebra approach. In Chapter 6, we present all known examples of $\mathcal{N} = 4, 5$ theories, and derive some new $\mathcal{N} = 4$ quiver gauge theories. In Chapter 7, we derive the $\mathcal{N}$=6 theories by decomposing the $\mathcal{N}=5$ fields and the symplectic 3-algebra properly. The $\mathcal{N}=8$ BLG theory is derived as a special case of the $\mathcal{N}$=6 theory. We also derive all known examples of the $\mathcal{N}=6,8$ theories by specifying the structure constants of the 3-brackets. In section 7.3, the $\mathcal{N}=6$ theories are also constructed by requiring that the supersymmetry transformations are closed on-shell. In Chapter 8, we rederive a super Lie algebra which can be used to realize the hermitian 3-algebra and the Nambu 3-algebra. By using the super-Lie-algebra realization of 3-algebras, the ordinary Lie algebra constructions of the $\mathcal{N}=6,8$ theories are rederived. In section 8.2, we propose a quantization scheme for the 3-brackets. The last chapter is devoted to conclusions. Our convention and useful identities are given in Appendix A. We verify the $Sp(4)$ global symmetry of the $\mathcal{N}=5$ bosonic potential in Appendix B. In Appendix C, we present some explicit examples of the $\mathcal{N}=5,6$ theories.

# CHAPTER 2

# SYMPLECTIC THREE-ALGEBRAS

A 3-algebra is a complex vector space equipped with a 3-bracket, mapping three vectors to one vector [40]:

$$[T_I, T_J; T_K] = f_{IJK}{}^L T_L, \tag{2.1}$$

where $T_I$ $(I = 1, 2, \cdots, M)$ is a set of generators. The set of complex numbers $f_{IJK}{}^L$ are called the structure constants. We define the global transformation of a field $X$ valued in this 3-algebra ($X = X^K T_K$) as [14]:

$$\delta_{\tilde{\Lambda}} X = \Lambda^{IJ}[T_I, T_J; X], \tag{2.2}$$

where the parameter $\Lambda^{IJ}$ is independent of spacetime coordinate. (The symmetry transformation (2.2) will be gauged later). The above equation is the natural generalization of $\delta_\Lambda X = \Lambda^a[T_a, X]$ in an ordinary Lie 2-algebra. For an ordinary Lie 2-algebra, the Jacobi identity is equivalent to

$$\delta_\Lambda([X, Y]) = [\delta_\Lambda X, Y] + [X, \delta_\Lambda Y]. \tag{2.3}$$

That is, $\delta_\Lambda X = \Lambda^a[T_a, X]$ must act as a derivative. Analogously, for (2.2) to be a symmetry, one has to require that it acts as a derivative [14]:

$$\delta_{\tilde{\Lambda}}([X, Y; Z]) = [\delta_{\tilde{\Lambda}} X, Y; Z] + [X, \delta_{\tilde{\Lambda}} Y; Z] + [X, Y; \delta_{\tilde{\Lambda}} Z], \tag{2.4}$$

where $Y = Y^N T_N$ and $Z = Z^K T_K$. Canceling $\Lambda^{IJ}, X^M, Y^N$ and $Z^K$ from both sides, we obtain the following fundamental identity (FI) satisfied by the generators:

$$[T_I, T_J; [T_M, T_N; T_K]] = [[T_I, T_J; T_M], T_N; T_K] + [T_M, [T_I, T_J; T_N]; T_K] + [T_M, T_N; [T_I, T_J; T_K]]. \tag{2.5}$$

The FI is a generalization of the Jacobi identity of an ordinary Lie algebra. Combining the three-bracket (2.1) and the FI (2.5), we find that the FI satisfied by the structure constants is

$$f_{MNK}{}^O f_{IJO}{}^L = f_{IJM}{}^O f_{ONK}{}^L + f_{IJN}{}^O f_{MOK}{}^L + f_{IJK}{}^O f_{MNO}{}^L. \tag{2.6}$$

To define a symplectic 3-algebra, we introduce a symplectic bilinear form into the 3-algebra:

$$\omega(X,Y) = \omega_{IJ} X^I Y^J. \tag{2.7}$$

We denote the inverse of the antisymmetric tensor $\omega_{IJ}$ as $\omega^{IJ}$. [1] The existence of the inverse implies that a 3-algebra index $I$ must run from 1 to $M = 2L$. We will use $\omega_{IJ}$ and $\omega^{IJ}$ to lower or raise 3-algebra indices; for instance, $f_{IJKL} \equiv \omega_{LM} f_{IJK}{}^M$. The symplectic bilinear form must be invariant under an arbitrary global transformation:

$$\begin{aligned} \delta_{\tilde{\Lambda}}(\omega_{IJ} X^I Y^J) &= \Lambda^{LM}(f_{LMI}{}^K \omega_{KJ} + f_{LMJ}{}^K \omega_{IK}) X^I Y^J \\ &= 0. \end{aligned} \tag{2.8}$$

It turns out that the structure constants must be *symmetric* in the last two indices:

$$f_{LMIJ} = f_{LMJI}. \tag{2.9}$$

From point of view of ordinary Lie group, the infinitesimal matrices

$$\tilde{\Lambda}^K{}_I \equiv \Lambda^{LM} f_{LM}{}^K{}_I \tag{2.10}$$

must form the Lie algebra $Sp(2L, \mathbb{C})$. We call the 3-algebra defined by the above equations a symplectic 3-algebra.

Since the 3-algebra is also a complex vector space, one can define a hermitian bilinear form

$$h(X,Y) = X^{*I} Y^I \tag{2.11}$$

(with $X^{*I}$ the complex conjugate of $X^I$), which is naturally positive-definite and will be used to construct the Lagrangians. The hermitian bilinear form is also required to be invariant under the global transformation:

$$\begin{aligned} \delta_{\tilde{\Lambda}}(X^{*I} Y^I) &= (\Lambda^{*LM} f^*_{LMI}{}^K + \Lambda^{LM} f_{LMK}{}^I) X^{*I} Y^K \\ &= 0. \end{aligned} \tag{2.12}$$

To solve the above equation, we assume that the parameter $\Lambda^{LM}$ is hermitian: $\Lambda^{*LM} = \Lambda_{ML}$. Since it also carries two symplectic 3-algebra indices, it obeys the natural real-

[1] In order to close the $\mathcal{N} \geq 4$ super Pioncare algebras, one must introduce the antisymmetric tensor into the theories (see sections 3.2 and 4.2).

ity condition $\Lambda^{*LM} = \omega_{LI}\omega_{MJ}\Lambda^{IJ}$. These two equations imply that the parameter is symmetric, i.e., $\Lambda_{ML} = \Lambda_{LM}$. In summary, we have

$$\Lambda^{*LM} = \Lambda_{ML} = \Lambda_{LM}. \tag{2.13}$$

Now since the parameter $\Lambda^{IJ}$ is symmetric, re-examining the global transformation (2.2) leads us to require that the structure constants are symmetric in the first two indices:

$$f_{IJKL} = f_{JIKL}. \tag{2.14}$$

With Eq. (2.13) and (2.14), we find that Eq. (2.12) can be satisfied if we impose the following reality condition on the structure constants:

$$f^{*}_{LMIK} = f^{MLKI} \quad \text{or} \quad f^{*L}{}_{M}{}^{I}{}_{K} = f^{M}{}_{L}{}^{K}{}_{I}. \tag{2.15}$$

Now both the symplectic bilinear form (2.7) and the hermitian bilinear (2.11) form are invariant under the global transformation (2.2). So from point of view of ordinary Lie group, the symmetry group generated by the 3-algebra transformations (2.2) is nothing but $Sp(2L)$, which is the intersection of $U(2L)$ and $Sp(2L,\mathbb{C})$.

By using the FI (2.6), one can prove that the structure constants $f_{IJK}{}^{L}$ are also preserved under the global symmetry transformations [38]:

$$\begin{aligned}
\delta_{\tilde\Lambda} f_{MNK}{}^{L} &= -\tilde\Lambda^{O}{}_{M} f_{ONK}{}^{L} - \tilde\Lambda^{O}{}_{N} f_{MOK}{}^{L} - \tilde\Lambda^{O}{}_{K} f_{MNO}{}^{L} + \tilde\Lambda^{L}{}_{O} f_{MNK}{}^{O} \\
&= \Lambda^{IJ}(-f_{IJM}{}^{O} f_{ONK}L - f_{IJN}{}^{O} f_{MOK}{}^{L} - f_{IJK}{}^{O} f_{MNO}{}^{L} + f_{IJO}{}^{L} f_{MNK}{}^{O}) \\
&= 0,
\end{aligned} \tag{2.16}$$

where we have used the FI (2.6) in the second line. In other words, Eq. (2.16) is equivalent to the FI (2.6). Thus we can use $\omega_{IJ}$ and $f_{IJK}{}^{L}$ to construct invariant Lagrangians, when the symmetry is gauged.

Later we will see, to enhance the super-symmetry from $\mathcal{N} = 1$ to $\mathcal{N} = 5$, we will require the 3-bracket to satisfy an additional constraint condition:

$$\omega([T_I, T_{(J}; T_K], T_{L)}) = 0, \tag{2.17}$$

or simply $f_{I(JKL)} = 0$. Combining Eq. (2.17) with (2.9) and (2.14), we have that $f_{(IJK)L} = 0$ and $f_{IJKL} = f_{KLIJ}$. In summary, the structure constants $f_{IJKL}$ enjoy the symmetry properties

$$f_{IJKL} = f_{JIKL} = f_{JILK} = f_{KLIJ}. \tag{2.18}$$

# CHAPTER 3

# $\mathcal{N}=5$ THEORIES AND 3-ALGEBRAS

In this chapter, we will generalize Giaotto and Witten's idea and method [33] to enhance the super-symmetry from $\mathcal{N}=1$ to $\mathcal{N}=5$ [44]. We will work in a three-algebraic framework. The closure of the $\mathcal{N}=5$ algebra will be checked explicitly [40].

## 3.1 $\mathcal{N}=5$ Theories in Terms of 3-Algebras

Let us first explain the mechanism for supersymmery enhancement. We assume that the $\mathcal{N}=1$ superfields for the matter fields are 3-algebra valued (our notation and convention are summarized in Appendix A):

$$\Phi^I_A = Z^I_A + i\theta\gamma_A{}^B\psi^I_B - \frac{i}{2}\theta^2 F^I_A, \tag{3.1}$$

where $I$ is a 3-algebra index, $A, B$ are $Sp(4)\cong SO(5)$ indices ($A,B=1,...,4$); and $\gamma_A{}^B$ is a Hermitian $SO(5)\equiv Sp(4)$ gamma matrix, satisfying $\gamma_A{}^B\gamma_B{}^C=\delta_A{}^C$. [1] The superfield $\Phi$ satisfies the reality condition:

$$\bar\Phi^A_I = \Phi^{\dagger I}_A = \omega^{AB}\omega_{IJ}\Phi^J_B. \tag{3.2}$$

The purpose for introducing the gamma matrix into the second term of (3.1) is the following: after we promote the supersymmetry from $\mathcal{N}=1$ to $\mathcal{N}=5$, we want the supercharges and the matter fields to transform as the **5** and **4** of $Sp(4)$, respectively, with the gamma matrix being the couplings.

Despite the fact that $\Phi^I_A$ carries an $Sp(4)$ index, it is still an $\mathcal{N}=1$ superfields in that it just depends on one copy of fermionic coordinates $\theta^\alpha$. Generally speaking, if we use (3.1) to construct an $\mathcal{N}=1$ CSM theory, the Yukawa couplings will contain the gamma

[1] Generally $\gamma_A{}^B\equiv c_m\gamma^{mB}_A$ ($m=1,...,5$), where $\gamma^{mB}_A$ are the $SO(5)$ gamma matrices (see Appendix A.4), and $c_m$ real coefficients. We normalize the parameters $c_m$ so that $\delta^{mn}c_mc_n=1$. The nonuniqueness of this gamma matrix is exactly what is allowed by the R-symmetry $SO(5)$.

matrix $\gamma_A{}^B$, which is not $Sp(4)$ invariant. [2] As a result, the CSM theory is generally not $Sp(4)$ invariant. However, we are be able to remove the gamma matrix $\gamma_A{}^B$ from the theory by adjusting the superspace couplings. The resulting theory then has an $Sp(4)$ global symmetry, which does *not* commute with the $\mathcal{N}=1$ supersymmetry. Namely the supercharge transforms nontrivially under the $Sp(4)$ global symmetry group. More precisely, the supercharges transform in the vector representation of $SO(5)$ or **5** of $Sp(4)$. As a result, the supersymmetry gets enhanced from $\mathcal{N}=1$ to $\mathcal{N}=5$. We will explain this point in detail when we examine the supersymmetry transformations.

To construct the $\mathcal{N}=1$ CSM theory, we first gauge the global symmetry transformation (2.2). We define the gauge transformation of the superfield $\Phi^I$ as

$$\delta_{\tilde{\Lambda}}\Phi^I_A = \Lambda^{KL}f_{KL}{}^I{}_J\Phi^J_A = \tilde{\Lambda}^I{}_J\Phi^J_A, \tag{3.3}$$

where the parameter $\Lambda^{KL}$ is a superfield, depending on the coordinates of the superspace. We then define the covariant derivatives as

$$(D_\alpha)^I{}_J = \mathscr{D}_\alpha\delta^I{}_J + \tilde{\Gamma}_\alpha{}^I{}_J \quad \text{and} \quad (D_\mu)^I{}_J = \partial_\mu\delta^I{}_J + \tilde{\Gamma}_\mu{}^I{}_J, \tag{3.4}$$

where $\mathscr{D}_\alpha$ is the supercovariant derivative, defined by Eq. (A.9). In accordance with our basic definition (2.2), it is natural to assume that the superconnections take the following forms

$$\tilde{\Gamma}_\alpha{}^I{}_J \equiv \Gamma^{KL}_\alpha f_{KL}{}^I{}_J \quad \text{and} \quad \tilde{\Gamma}_\mu{}^I{}_J \equiv \Gamma^{KL}_\mu f_{KL}{}^I{}_J, \tag{3.5}$$

transforming as [3]

$$\delta_{\tilde{\Lambda}}\tilde{\Gamma}_\alpha{}^I{}_J = -D_\alpha\tilde{\Lambda}^I{}_J \quad \text{and} \quad \delta_{\tilde{\Lambda}}\tilde{\Gamma}_\mu{}^{\mathrm{I}}{}_{\mathrm{J}} = -\mathrm{D}_\mu\tilde{\Lambda}^{\mathrm{I}}{}_{\mathrm{J}}, \tag{3.6}$$

respectively. In the Wess-Zumino gauge, the superconnection $\tilde{\Gamma}_\alpha$ takes the form

$$\begin{aligned}\tilde{\Gamma}_\alpha{}^I{}_J &= i\theta^\beta\tilde{A}_{\alpha\beta}{}^I{}_J + \theta^2\tilde{\chi}_\alpha{}^I{}_J \\ &= (i\theta^\beta A^{KL}_{\alpha\beta} + \theta^2\chi^{KL}_\alpha)f_{KL}{}^I{}_J,\end{aligned} \tag{3.7}$$

where $\tilde{\chi}_\alpha{}^I{}_J$ is superpartner of the gauge field $\tilde{A}_{\alpha\beta}{}^I{}_J$. In accordance with (2.13), we assume that $A^{KL}_{\alpha\beta}$ and $\chi^{KL}_\alpha$ are hermitian and symmetric in $KL$. The two superconnections (3.5)

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[2] With the standard definition $\Sigma_A{}^B \equiv \frac{1}{2}\omega_{mn}\Sigma^{mnB}_A$, where $\Sigma^{mn} = \frac{1}{4}[\gamma^m,\gamma^n]$, we note that

$$\delta\gamma_A{}^B \equiv \Sigma_A{}^C\gamma_C{}^B - \Sigma_C{}^B\gamma_A{}^C = \omega_{mn}c^n\gamma^{mB}_A.$$

Thus, $\gamma_A{}^B$ is *not* $Sp(4)$ invariant.

[3] In this section, we define a general tilde field $\tilde{\Psi}$ as $\tilde{\Psi}^I{}_J \equiv \Psi^{KL}f_{KL}{}^I{}_J$, where $\Psi^{KL}$ can be a superfield or an ordinary field.

should not be independent, since there is only one gauge symmetry. Actually, imposing the conventional constraint [41]

$$\{D_\alpha, D_\beta\} = 2iD_{\alpha\beta} \tag{3.8}$$

determines the vector superconnection:

$$\tilde{\Gamma}_{\alpha\beta}{}^I{}_J = \tilde{A}_{\alpha\beta}{}^I{}_J - i\theta_\alpha \tilde{\chi}_\beta{}^I{}_J - i\theta_\beta \tilde{\chi}_\alpha{}^I{}_J + \frac{i}{2}\theta^2 \tilde{F}_{\alpha\beta}{}^I{}_J, \tag{3.9}$$

where the field strength is defined as

$$\tilde{F}_{\alpha\beta}{}^I{}_J = \frac{1}{2}(\partial_\alpha{}^\gamma \tilde{A}_{\gamma\beta}{}^I{}_J + \partial_\beta{}^\gamma \tilde{A}_{\gamma\alpha}{}^I{}_J) + \frac{1}{2}[\tilde{A}_\alpha{}^\gamma, \tilde{A}_{\gamma\beta}]^I{}_J; \quad \tilde{F}_{\mu\nu}{}^I{}_J = \frac{1}{2}(\gamma_{\mu\nu})^{\alpha\beta}\tilde{F}_{\alpha\beta}{}^I{}_J. \tag{3.10}$$

The superfield $\Gamma^{KL}_\mu = -\frac{1}{2}\gamma^{\alpha\beta}_\mu \Gamma^{KL}_{\alpha\beta}$ in Eq. (3.5) can be read off from Eq. (3.9) by rewriting the field strength as a product of a field and the structure constants:

$$\begin{aligned}
\tilde{F}_{\alpha\beta}{}^I{}_J &= \frac{1}{2}[\partial_\alpha{}^\gamma A^{KL}_{\gamma\beta} + \partial_\beta{}^\gamma A^{KL}_{\gamma\alpha} + (\tilde{A}_\alpha{}^\gamma)^L{}_M A^{MK}_{\gamma\beta} + (\tilde{A}_\beta{}^\gamma)^K{}_M A^{ML}_{\gamma\alpha}] f_{KL}{}^I{}_J \\
&\equiv F^{KL}_{\alpha\beta} f_{KL}{}^I{}_J.
\end{aligned} \tag{3.11}$$

In the first line we have used the FI (2.6).

To be self-consistent, the covariant derivative $D_\alpha$ must satisfy the Jacobi identity:

$$[D_\alpha, \{D_\beta, D_\gamma\}] + [D_\beta, \{D_\gamma, D_\alpha\}] + [D_\gamma, \{D_\alpha, D_\beta\}] = 0. \tag{3.12}$$

The Jacobi identity can be solved by introducing a superfield strength $\tilde{\mathcal{W}}_\alpha$ [41]:

$$[D_\alpha, D_{\beta\gamma}] = i\epsilon_{\alpha\beta}\tilde{\mathcal{W}}_\gamma + i\epsilon_{\alpha\gamma}\tilde{\mathcal{W}}_\beta. \tag{3.13}$$

By direct calculation, we obtain

$$\begin{aligned}
\tilde{\mathcal{W}}_\alpha{}^I{}_J &= \tilde{\chi}_\alpha{}^I{}_J + \theta^\beta \tilde{F}_{\alpha\beta}{}^I{}_J - \frac{i}{2}\theta^2 (D_\alpha{}^\beta \tilde{\chi}_\beta)^I{}_J \\
&= [\chi^{KL}_\alpha + \theta^\beta F^{KL}_{\alpha\beta} - \frac{i}{2}\theta^2 (D_\alpha{}^\beta \chi_\beta)^{KL}] f_{KL}{}^I{}_J \\
&\equiv \mathcal{W}^{KL}_\alpha f_{KL}{}^I{}_J,
\end{aligned} \tag{3.14}$$

with

$$(D_\alpha{}^\beta \chi_\beta)^{KL} f_{KL}{}^I{}_J \equiv [\partial_\alpha{}^\beta \chi^{KL}_\beta + (\tilde{A}_\alpha{}^\beta)^L{}_M \chi^{MK}_\beta + (\tilde{A}_\alpha{}^\beta)^K{}_M \chi^{MJ}_\beta] f_{KL}{}^I{}_J. \tag{3.15}$$

In deriving the above equation, we have used the FI (2.6) again. Here we would like to make one comment on the relation between the FI (2.6) and the anticommutator (3.8) and the Jacobi identity (3.12). Without consulting the FI, one would not be able to derive Eq. (3.11) and write $\tilde{\Gamma}_{\alpha\beta}{}^I{}_J$ as $\Gamma^{KL}_{\alpha\beta} f_{KL}{}^I{}_J$. This would be inconsistent with our assumption (3.5) or the basic definition (2.2). Similarly, the superfield strength would not take the form $\tilde{\mathcal{W}}_\alpha{}^I{}_J = \mathcal{W}^{KL}_\alpha f_{KL}{}^I{}_J$ without the FI (see Eq. (3.14)). Recall that the vector superconnection and the superfield strength are defined through (3.8) and (3.12), respectively. So, had we not introduced the FI in section 2, we would have to introduce the FI in this subsection for making the 3-bracket (2.2) consistent with (3.8) and (3.12).

After gauging the symmetry (2.2) in the superspace, we are ready to construct an $\mathcal{N} = 1$ CSM theory. A general $\mathcal{N} = 1$ CSM theory consists of three parts: $\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{CS}} + \mathcal{L}_W$, where $\mathcal{L}_{\text{kin}}$ is the Lagrangian of the kinetic terms of the matter fields, $\mathcal{L}_{\text{CS}}$ the Chern-Simons term and $\mathcal{L}_W$ the superpotential. The first part $\mathcal{L}_{\text{kin}}$ is standard:

$$\begin{aligned} \mathcal{L}_{\text{kin}} &= \frac{1}{8}\int d^2\theta D^\alpha \bar{\Phi}^A_I D_\alpha \Phi^I_A \\ &= \frac{1}{2}(-D_\mu \bar{Z}^A_I D^\mu Z^I_A + i\bar{\psi}^A_I \gamma_\mu D^\mu \psi^I_A + 2i f_{IJKL}\gamma_B{}^A \bar{\psi}^{BK}\chi^{IJ} Z^L_A + \bar{F}^A_I F^I_A). \end{aligned} \tag{3.16}$$

The covariant derivatives are given by

$$D_\mu Z^A_I = \partial_\mu Z^A_I - \tilde{A}_\mu{}^J{}_I Z^A_J, \tag{3.17}$$

$$D_\mu Z^I_A = \partial_\mu Z^I_A + \tilde{A}_\mu{}^I{}_J Z^J_A. \tag{3.18}$$

We propose the Chern-Simons term as

$$\begin{aligned} \mathcal{L}_{\text{CS}} &= \frac{1}{8}\int d^2\theta[-i f_{IJKL}\Gamma^{\alpha IJ}\mathcal{W}^{KL}_\alpha + \frac{1}{3} f_{IJK}{}^O f_{OLMN}\Gamma^{\alpha IJ}\Gamma^{\beta KL}\Gamma^{MN}_{\alpha\beta}] \\ &= \frac{1}{2}\epsilon^{\mu\nu\lambda}(f_{IJKL}A^{IJ}_\mu \partial_\nu A^{KL}_\lambda + \frac{2}{3} f_{IJK}{}^O f_{OLMN} A^{IJ}_\mu A^{KL}_\nu A^{MN}_\lambda) + \frac{i}{2} f_{IJKL}\chi^{\alpha IJ}\chi^{KL}_\alpha. \end{aligned} \tag{3.19}$$

The first part of the second line is precisely the 'twisted' Chern-Simons term in Ref. [40], while the gaugino $\chi$ is just an auxiliary field, whose equation of motion is

$$\chi^{\alpha IJ} = -\gamma_B{}^A \psi^{\alpha B(I} Z^{J)}_A. \tag{3.20}$$

Substituting it into (3.16) and (3.19) gives a Yukawa coupling:

$$-\frac{i}{2} Z^I_A Z^J_B \psi^K_C \psi^L_D f_{IKJL}\gamma^{AC}\gamma^{BD}. \tag{3.21}$$

Note that this term is not $Sp(4)$ invariant, because the gamma matrix is *not* $Sp(4)$ invariant (see footnote 2).

Let us now consider the superpotential $W(\Phi)$. It must satisfy two conditions. First, for conformal invariance, the superpotential must be homogeneous and quartic in $\Phi$;

schematically, $W(\Phi) \sim \Phi\Phi\Phi\Phi$. Secondly, after combining (3.21) with the Yukawa terms arising from $W(\Phi)$, the final expression must be $Sp(4)$ invariant. Before proposing $W(\Phi)$, it is useful to look at the structure of (3.21): it contains $\gamma^{AC}\gamma^{BD}$. The essential observation is that $\gamma^{[AC}\gamma^{BD]}$ has to be proportional to the totally antisymmetric (invariant) tensor $\varepsilon^{ABCD}$, since this tensor is unique in $Sp(4)$. The precise expression is

$$\begin{aligned} -\varepsilon^{ABCD} &= \gamma^{AC}\gamma^{BD} - \gamma^{BC}\gamma^{AD} + \gamma^{BA}\gamma^{CD} \\ &= \omega^{AB}\omega^{CD} - \omega^{AC}\omega^{BD} + \omega^{AD}\omega^{BC}. \end{aligned} \tag{3.22}$$

Namely, our problem may be solved if the final expression for (3.21) plus the Yukawa terms arising from $W(\Phi)$ are somehow related to (3.22). So we are inspired to propose the following superpotential

$$W(\Phi) = \frac{1}{4}(g_{IJKL}\omega^{AB}\omega^{CD}\Phi_A^I\Phi_B^J\Phi_C^K\Phi_D^L + \tilde{g}_{IJKL}\gamma^{AB}\gamma^{CD}\Phi_A^I\Phi_B^J\Phi_C^K\Phi_D^L), \tag{3.23}$$

where the 3-algebra tensor $g$ satisfies $g_{IJKL} = -g_{JIKL} = -g_{IJLK} = g_{KLIJ}$, and $\tilde{g}$ has the same symmetry properties. We require that the tensors $g$ and $\tilde{g}$ are gauge invariant. This implies that $g$ and $\tilde{g}$ can be expressed in terms of $\omega_{IJ}$ and $f_{IJKL}$, the only two gauge invariant quantities. After carrying out the Berezin integration $\frac{i}{2}\int d^2\theta W(\Phi)$, we obtain

$$\begin{aligned} \mathcal{L}_W \;=\; & -\frac{i}{2}Z_A^I Z_B^J \psi_C^K \psi_D^L (g_{IJKL}\omega^{AB}\omega^{CD} + 2g_{IKJL}\gamma^{AC}\gamma^{BD} + \tilde{g}_{IJKL}\gamma^{AB}\gamma^{CD} \\ & +2\tilde{g}_{IKJL}\omega^{AC}\omega^{BD}) - (g_{IJKL}\omega^{AB}\omega^{CD} + \tilde{g}_{IJKL}\gamma^{AB}\gamma^{CD})Z_B^J Z_C^K Z_D^L F_A^I. \end{aligned} \tag{3.24}$$

The first and last term of the first line are already $Sp(4)$ invariant. Combining the middle two terms of the first line with (3.21) gives

$$-\frac{i}{2}Z_A^I Z_B^J \psi_C^K \psi_D^L [(2g_{IKJL} + f_{IKJL})\gamma^{AC}\gamma^{BD} + \tilde{g}_{IJKL}\gamma^{AB}\gamma^{CD}]. \tag{3.25}$$

Since we wish to use Eq. (3.22), we first have to antisymmetrize $AB$ in the expression $\gamma^{AC}\gamma^{BD}$. Equivalently, we have to set the part proportional to $Z^{(I}_{(A}Z^{J)}_{B)}$ to be zero:

$$g_{IKJL} + g_{JKIL} + \frac{1}{2}f_{IKJL} + \frac{1}{2}f_{JKIL} = 0. \tag{3.26}$$

Now the remaining part of (3.25) is antisymmetric in $AB$:

$$\frac{i}{2}Z_A^I Z_B^J \psi_C^K \psi_D^L [(2g_{IKJL} + f_{IKJL})\gamma^{C[A}\gamma^{B]D} - \tilde{g}_{IJKL}\gamma^{AB}\gamma^{CD}]. \tag{3.27}$$

It can be seen that if we set

$$\tilde{g}_{IJKL} = -\frac{1}{2}(g_{IKJL} - g_{JKIL} + \frac{1}{2}f_{IKJL} - \frac{1}{2}f_{JKIL}) \tag{3.28}$$

and apply the key identity (3.22), then Eq. (3.27) becomes

$$\frac{i}{2} Z^I_A Z^J_B \psi^K_C \psi^L_D \tilde{g}_{IJKL} (\omega^{AB}\omega^{CD} - \omega^{AC}\omega^{BD} + \omega^{AD}\omega^{BC}). \tag{3.29}$$

Now Eq. (3.29) is manifestly $Sp(4)$ invariant. However we still need to solve (3.26) and (3.28) in terms of $f_{IJKL}$ and $\omega_{IJ}$. An equation similar to (3.26) is first derived by GW [33]:

$$g_{IKJL} + g_{JKIL} + \frac{3}{4} k_{mn} \tau^m_{IK} \tau^n_{JL} + \frac{3}{4} k_{mn} \tau^m_{JK} \tau^n_{IL} = 0, \tag{3.30}$$

where the set of matrices $\tau^m_{IK}$ is in the fundamental representation of $Sp(2L)$ or its subalgebra, and $k_{mn}$ is the Killing-Cartan metric. Although the ($\mathcal{N} = 4$) GW theory is not an $\mathcal{N} = 5$ theory, the similarity between (3.26) and (3.30) strongly suggests that $f_{IJKL}$ can be specified as $k_{mn}\tau^m_{IJ}\tau^n_{KL}$ (up to an unimportant constant). This is indeed the case: the FI (2.6) does admit an explicit solution in terms of the tensor product $f_{IJKL} = k_{mn}\tau^m_{IJ}\tau^n_{KL}$. It is straightforward to verify that $f_{IJKL} = k_{mn}\tau^m_{IJ}\tau^n_{KL}$ satisfy the FI (2.6). This solution is first found by Gustavsson by converting the FI into two independent commutators of ordinary Lie algebra [15]. Later we will discuss the relations between the 3-algebra and the ordinary Lie algebra in detail. Eq. (3.26) can be easily solved by adopting a method in Ref. [33]. Summing (3.26) over cyclic permutations of $IKJ$ gives

$$f_{(IKJ)L} = 0, \quad \text{or} \quad f_{I(KJL)} = 0. \tag{3.31}$$

This is precisely (2.17), as we stated earlier. The above equation is also derived by requiring that the $\mathcal{N} = 5$ supersymmetry transformations are closed on-shell [40]. Eq. (3.26) is solved by setting

$$g_{IKJL} = \frac{1}{6}(f_{IJKL} - f_{ILKJ}). \tag{3.32}$$

Substituting (3.32) into (3.28), we obtain

$$\tilde{g}_{IJKL} = \frac{1}{3}(f_{ILJK} - f_{IKJL}). \tag{3.33}$$

Substituting (3.33) into (3.29), then combining (3.29) with the first and the last term of the first line of (3.24), we reach the final expression for all Yukawa terms:

$$-\frac{i}{2}\omega^{AB}\omega^{CD} f_{IJKL}(Z^I_A Z^K_B \psi^J_C \psi^L_D - 2 Z^I_A Z^K_D \psi^J_C \psi^L_B). \tag{3.34}$$

Finally we integrate out the auxiliary field $F^I_A$ appearing in (3.16) and (3.24):

$$\bar{F}^A_I = \frac{1}{3} f_{IKLJ}\omega^{BC}\omega^{AD} Z^K_B Z^L_C Z^J_D - \frac{2}{3} f_{IKLJ}\gamma^{BC}\gamma^{AD} Z^K_B Z^L_C Z^J_D. \tag{3.35}$$

Now it is straightforward to calculate the bosonic potential:

$$\begin{aligned} -\frac{1}{2}\bar{F}^A_I F^I_A \quad = \quad & \frac{1}{18} f_{IJKO} f^O{}_{LMN}(-\omega^{AC}\omega^{BE}\omega^{DF} + 2\omega^{AC}\gamma^{BE}\gamma^{DF} \\ & +2\omega^{DF}\gamma^{AC}\gamma^{BE} - 4\omega^{BE}\gamma^{AC}\gamma^{DF}) Z^I_A Z^J_B Z^K_C Z^L_D Z^M_E Z^N_F. \end{aligned} \tag{3.36}$$

Note that $V = \frac{1}{2}\bar{F}^A_I F^I_A$ is positive definite, though it is not manifestly $Sp(4)$ invariant due to the presence of the gamma matrix. However, by taking advantage of the key identity (3.22) and the constraint condition $f_{(IJK)L} = 0$, we are able to prove that (3.36) is indeed $Sp(4)$ invariant (see Appendix B). The final expression for the bosonic potential is

$$V = -\frac{1}{60}(2f_{IJK}{}^O f_{OLMN} - 9f_{KLI}{}^O f_{ONMJ} + 2f_{IJL}{}^O f_{OKMN}) Z^N_A Z^{AI} Z^J_B Z^{BK} Z^L_C Z^{CM}. \tag{3.37}$$

In summary, the full Lagrangian in terms of the symplectic 3-algebra is given by

$$\begin{aligned} \mathcal{L} \quad = \quad & \frac{1}{2}(-D_\mu \bar{Z}^A_I D^\mu Z^I_A + i\bar{\psi}^A_I \gamma_\mu D^\mu \psi^I_A) \\ & -\frac{i}{2}\omega^{AB}\omega^{CD} f_{IJKL}(Z^I_A Z^K_B \psi^J_C \psi^L_D - 2Z^I_A Z^K_D \psi^J_C \psi^L_B) \\ & +\frac{1}{2}\epsilon^{\mu\nu\lambda}(f_{IJKL} A^{IJ}_\mu \partial_\nu A^{KL}_\lambda + \frac{2}{3} f_{IJK}{}^O f_{OLMN} A^{IJ}_\mu A^{KL}_\nu A^{MN}_\lambda) \\ & +\frac{1}{60}(2f_{IJK}{}^O f_{OLMN} - 9f_{KLI}{}^O f_{ONMJ} + 2f_{IJL}{}^O f_{OKMN}) Z^N_A Z^{AI} Z^J_B Z^{BK} Z^L_C Z^{CM}. \end{aligned} \tag{3.38}$$

This Lagrangian is exactly the same as the $\mathcal{N} = 5$ Lagrangian derived by requiring that the supersymmetry transformations are closed on-shell [40]. Using the reality condition (2.15), one can recast the potential term into the following form:

$$V = \frac{2}{15}(\Upsilon^L_{ABC})^* \Upsilon^L_{ABC}, \tag{3.39}$$

where

$$\Upsilon^L_{ABC} \equiv f_{IJK}{}^L (Z^I_A Z^J_B Z^K_C + \frac{1}{4}\omega_{BC} Z^I_A Z^J_D Z^{DK}). \tag{3.40}$$

Now the potential term is manifestly positive definite.

Let us consider the supersymmetry transformations. The $\mathcal{N} = 1$ supersymmetry transformation of the scalar field is

$$\delta_Q Z^I_A = i\epsilon^\alpha \gamma_A{}^B \psi^I_{\alpha B}. \tag{3.41}$$

On the other hand, the action (3.38) is invariant under the $Sp(4)$ global symmetry transformation

$$\delta_R Z^I_A = \Sigma_A{}^B Z^I_B \quad , \quad \delta_R \psi^I_A = \Sigma_A{}^B \psi^I_B. \tag{3.42}$$

Therefore one can consider the commutator of $\delta_R$ and $\delta_Q$:

$$[\delta_R, \delta_Q] Z^I_A = i\epsilon^\alpha (\gamma_A{}^B \Sigma_B{}^C - \Sigma_A{}^B \gamma_B{}^C)\psi^I_{\alpha C}. \tag{3.43}$$

So the $\mathcal{N} = 1$ supersymmetry does *not* commute with the $Sp(4)$ global symmetry. Since the matrix $\gamma_A{}^B$ contains four independent real parameters, equation (3.43) suggests that there are other 4 independent $\mathcal{N} = 1$ supersymmetries. Therefore one may promote the $\mathcal{N} = 1$ supersymmetry (3.41) to $\mathcal{N} = 5$:

$$\delta Z^I_A = i\epsilon_A{}^{B\alpha} \psi^I_{B\alpha}, \tag{3.44}$$

where the parameter $\epsilon_A{}^{B\alpha} = \epsilon^\alpha_m \gamma^{mB}_A$. One may apply the same argument to the supersymmetry transformations of the fermionic and gauge fields. In summary, we have the following supersymmetry transformations:

$$\begin{aligned}
\delta Z^I_A &= i\epsilon_A{}^{B\alpha}\psi^I_{B\alpha}, \\
\delta\psi^I_{A\alpha} &= (\gamma^\mu)_\alpha{}^\beta D_\mu Z^I_B \epsilon^B{}_{A\beta} + \frac{1}{3} f^I{}_{JKL}\omega^{BC} Z^J_B Z^K_C Z^L_D \epsilon^D{}_{A\alpha} - \frac{2}{3} f^I{}_{JKL}\omega^{BD} Z^J_C Z^K_D Z^L_A \epsilon^C{}_{B\alpha}, \\
\delta \tilde{A}_\mu{}^K{}_L &= i\epsilon^{AB\alpha}(\gamma_\mu)_\alpha{}^\beta \psi^J_{B\beta} Z^I_A f_{IJ}{}^K{}_L,
\end{aligned} \tag{3.45}$$

where the parameter $\epsilon^{AB}$ is antisymmetric in $AB$, satisfying

$$\begin{aligned}
&\omega_{AB}\epsilon^{AB} = 0, \\
&\epsilon^*_{AB} = \omega^{AC}\omega^{BD}\epsilon_{CD}.
\end{aligned} \tag{3.46}$$

The supersymetry transformations are precisely the ones proposed in Ref. [40]. To verify the mechanism for enhancing the $\mathcal{N} = 1$ to $\mathcal{N} = 5$, it is best to check the closure of (3.45); this will be done in the next section. Later we will see that they are indeed closed on-shell, and the corresponding equations of motion can be derived from the Lagrangian (3.38). So the R-symmetry of the theories is $Sp(4)$.

## 3.2 Closure of the $\mathcal{N}=5$ Algebra

Following BL's strategy [38], we will derive the equations of motion by requiring that the supersymmetry transformations are closed on-shell. Let us first examine scalar supersymmetry transformation. By virtue of the identities in Appendix A.4, we find

$$[\delta_1,\delta_2]Z_A^I \;=\; v^\mu D_\mu Z_A^I - \frac{2}{3}f_{JLK}{}^I\Lambda^{KL}Z^{AJ} + \frac{2}{3}f_{KLJ}{}^I\Lambda^{KL}Z_A^J, \qquad (3.47)$$

where

$$v^\mu \;\equiv\; -\frac{i}{2}\bar\epsilon_2^{BD}\gamma^\mu\epsilon_{1BD}, \qquad (3.48)$$

$$\Lambda^{KL} \;\equiv\; -\frac{i}{2}Z_D^K Z_C^L(\bar\epsilon_1^{CE}\epsilon_{2E}{}^D - \bar\epsilon_2^{CE}\epsilon_{1E}{}^D) = \Lambda^{LK}, \qquad (3.49)$$

and the $\epsilon$ bilinear is symmetric in $CD$. While the first term of Eq. (3.47) is the gauge covariant translation, we have to impose some conditions on the structure constants so that the remaining terms add up to be a gauge transformation. (We will read off the parameter of the gauge transformation by looking the closure of the algebra on the gauge fields.)

We tentatively assume that the third term of Eq. (3.47) is proportional to the gauge transformation. So the second term of Eq. (3.47) should be also proportional to the gauge transformation. This leads us to impose an additional constraint condition on the structure constants:

$$\frac{1}{2}(f_{JLK}{}^I + f_{JKL}{}^I) = \frac{\alpha}{2}f_{KLJ}{}^I, \qquad (3.50)$$

where $\alpha$ is a constant, to be determined later. Now the second and third term of Eq. (3.47) can be combined as

$$\frac{1}{3}(-\alpha+2)f_{KLJ}{}^I\Lambda^{KL}Z_A^J, \qquad (3.51)$$

which should be the gauge transformation.

Let us now look at the gauge fields:

$$\begin{aligned}[\delta_1,\delta_2]\tilde A_\mu{}^I{}_J \;=\;& v^\nu\tilde F_{\nu\mu}{}^I{}_J - (D_\mu\Lambda^{KL})f_{KL}{}^I{}_J\\ &+v^\nu[\tilde F_{\mu\nu}{}^I{}_J - \varepsilon_{\mu\nu\lambda}(Z_A^K D^\lambda Z^{AL} - \frac{i}{2}\bar\psi^{BK}\gamma^\lambda\psi_B^L)f_{KL}{}^I{}_J]\\ &+\mathcal{O}(Z^4),\end{aligned} \qquad (3.52)$$

where the last term $\mathcal{O}(Z^4)$ is fourth order in the scalar fields $Z$. We recognize the second term of the first line as a gauge transformation

$$-(D_\mu\Lambda^{KL})f_{KL}{}^I{}_J = -D_\mu(\Lambda^{KL}f_{KL}{}^I{}_J) \qquad (3.53)$$

by a parameter $\tilde{\Lambda}^I{}_J = \Lambda^{KL} f_{KL}{}^I{}_J$, since the FI (2.6) or (2.16) implies that $D_\mu f_{KL}{}^I{}_J = 0$ [38]. In accordance with the parameter, now (3.51) must satisfy the following equation[4]:

$$\frac{1}{3}(-\alpha + 2) f_{KLJ}{}^I \Lambda^{KL} Z^J_A = \Lambda^{KL} f_{KL}{}^I{}_J Z^J_A. \tag{3.54}$$

This equation can be solved by setting $\alpha = -1$. In other words, Eq. (3.54) is solved if Eq. (3.50) can be written as

$$f_{(JKL)}{}^I = 0, \tag{3.55}$$

since one can prove that

$$f_{KLJ}{}^I = f_{KL}{}^I{}_J. \tag{3.56}$$

Now Eq. (3.47) becomes

$$[\delta_1, \delta_2] Z^I_A \quad = \quad v^\mu D_\mu Z^I_A + \tilde{\Lambda}^I{}_J Z^J_A, \tag{3.57}$$

as expected.

Following Gustavsson's approach [15], one can demonstrate that the FI (2.6) admits an explicit solution in terms of a tensor product: $f_{IJKL} = k_{mn} \tau^m_{IJ} \tau^n_{KL}$, where $k_{mn}$ is the Killing-Cartan metric of $Sp(2L)$, and $\tau^m_{IJ} = \omega_{IK} \tau^{mK}{}_J$. The matrix $\tau^{mK}{}_J$ is in the fundamental representation of $Sp(2L)$, and $\omega_{IK}$ is the $Sp(2L)$-invariant antisymmetric tensor. Now Eq. (3.55) implies that $k_{mn} \tau^m_{(IJ} \tau^n_{K)L} = 0$, which is first derived by GW [33]. In the GW theories, it is the key requirement for enhancing the $\mathcal{N} = 1$ supersymmetry to the $\mathcal{N} = 4$ supersymmetry.

By using the FI (2.6) and the symmetry conditions (2.18), one can prove that the last term of Eq. (3.52) vanishes:

$$\mathcal{O}(Z^4) = 0. \tag{3.58}$$

So the second line of Eq. (3.52) must be the equations of motion for the gauge fields:

$$\tilde{F}_{\mu\nu}{}^I{}_J - \varepsilon_{\mu\nu\lambda}(Z^K_A D^\lambda Z^{AL} - \frac{i}{2} \bar{\psi}^{BK} \gamma^\lambda \psi^L_B) f_{KL}{}^I{}_J = 0. \tag{3.59}$$

Now only the first line of Eq. (3.52) remains:

$$[\delta_1, \delta_2] \tilde{A}_\mu{}^I{}_J \quad = \quad v^\nu \tilde{F}_{\nu\mu}{}^I{}_J - D_\mu \tilde{\Lambda}^I{}_J, \tag{3.60}$$

which is the desired result.

[4]According to our convention, if $\delta_{\tilde{\Lambda}} Z^I_A = \tilde{\Lambda}^I{}_J Z^J_A$, we must set $\delta_{\tilde{\Lambda}} \tilde{A}_\mu{}^I{}_J = -D_\mu \tilde{\Lambda}^I{}_J$ so that $\delta_{\tilde{\Lambda}}(D_\mu Z^I_A) = \tilde{\Lambda}^I{}_J (D_\mu Z^J_A)$.

Finally we turn to the fermion supersymmetry transformation:

$$\begin{aligned}[\delta_1, \delta_2]\psi^I_A &= v^\mu D_\mu \psi^I_A + \tilde{\Lambda}^I{}_J \psi^J_A \\ &+ \frac{i}{2}(\bar{\epsilon}^{BC}_1 \epsilon_{2BA} - \bar{\epsilon}^{BC}_2 \epsilon_{1BA}) E^I_C \\ &- \frac{1}{2} v_\nu \gamma^\nu E^I_A, \end{aligned} \tag{3.61}$$

where

$$E^I_A = \gamma^\mu D_\mu \psi^I_A - f_{KLJ}{}^I Z^J_B Z^{BK} \psi^L_A + 2 f_{KLJ}{}^I Z^J_B Z^K_A \psi^{BL}. \tag{3.62}$$

Hence the equations of motion for fermionic fields are $E^I_A = 0$. The scalar equations of motion can be derived by taking the super-variation of the fermionic equations of motion:

$$\delta E^I_A = 0. \tag{3.63}$$

After Fierz transformation, we obtain two independent parts, containing $\gamma^\mu \epsilon_{BC}$ and $\epsilon_{BC}$, respectively. The part containing $\gamma^\mu \epsilon_{BC}$ merely implies the equations of motion for the gauge fields, so we will not write it down here. The part containing $\epsilon_{BC}$ reads

$$\left( \delta^{[C}_A F^{B]I} + G_A{}^{BCI} \right) \epsilon_{BC} = 0, \tag{3.64}$$

where

$$F^{BI} \equiv -D^2 Z^{BI} + i f_{KLJ}{}^I Z^{CJ} \bar{\psi}^{BK} \psi^L_C + \frac{1}{3} f_{MNL}{}^O f_{OKJ}{}^I Z^J_C Z^{CK} Z^L_D Z^{DM} Z^{BN}, \tag{3.65}$$

and

$$\begin{aligned} G_A{}^{BCI} \epsilon_{BC} \equiv & \Big[ i f_{KLJ}{}^I (\frac{3}{2} Z^{BL} \bar{\psi}^{CK} \psi^J_A + Z^K_A \bar{\psi}^{CJ} \psi^{BL}) + i f_{L[KJ]}{}^I Z^{BJ} \bar{\psi}^{CK} \psi^L_A \\ & + \frac{2}{3} (f_{MNL}{}^O f_{OKJ}{}^I + f_{KMJ}{}^O f_{ONL}{}^I + 2 f_{MJL}{}^O f_{ONK}{}^I) \\ & \times Z^J_D Z^{DK} Z^{CL} Z^{BM} Z^N_A \Big] \epsilon_{BC}. \end{aligned} \tag{3.66}$$

Since the parameters $\epsilon_{BC}$ are traceless, in the sense that $\omega^{BC}\epsilon_{BC} = \epsilon_B{}^B = 0$, Eq. (3.64) must be equivalent to the following traceless equation:

$$\delta^{[C}_A F^{B]I} - \frac{1}{4} \omega^{BC} F^I_A + G_A{}^{BCI} - \frac{1}{4} \omega^{BC} \omega_{DE} G_A{}^{EDI} = 0. \tag{3.67}$$

Contracting on $AC$ gives the scalar equations of motion:

$$F^{BI} + \frac{4}{5} G_A{}^{BAI} - \frac{1}{5} G^B{}_A{}^{AI} = 0. \tag{3.68}$$

After some simplification we obtain

$$
\begin{aligned}
0 = & -D^2 Z_I^B - i f_{IJK}{}^L (Z_L^B \bar{\psi}^{CK} \psi_C^J - 2 Z^{CK} \bar{\psi}_C^J \psi_L^B) \\
& - \frac{1}{5} (f_{IJK}{}^O f_{OLM}{}^N + f_{IJL}{}^O f_{OKM}{}^N + 3 f_{IJM}{}^O f_{KLO}{}^N - 3 f_{IJM}{}^O f_{OLK}{}^N) \\
& \times Z_A^J Z^{AK} Z_C^L Z^{CM} Z_N^B.
\end{aligned}
\tag{3.69}
$$

All the equations of motion can be derived as the Euler-Lagrangian equations from the action (3.38).

# CHAPTER 4

# $\mathcal{N}=4$ THEORIES AND SYMPLECTIC 3-ALGEBRAS

## 4.1 $\mathcal{N}=4$ Theories by Starting from $\mathcal{N}=5$ Theories

In this section, we will construct the $\mathcal{N}$=4 theories by decomposing the $\mathcal{N}=5$ supermultiplets and the symplectic 3-algebra properly and proposing a new superpotential term that preserves only $\mathcal{N}=4$ [44]. Let us first decompose the $\mathcal{N}=5$ superfields for matter fields into $\mathcal{N}=4$ superfields:

$$(\Phi^I_A)_{\mathcal{N}=5} = \begin{pmatrix} \Phi^a_A \\ \Phi^{a'}_{\dot A} \end{pmatrix} = \begin{pmatrix} Z^a_A \\ Z^{a'}_{\dot A} \end{pmatrix} + i \begin{pmatrix} 0 & \sigma_A{}^{\dot A} \\ \sigma^{\dagger}_{\dot A}{}^{A} & 0 \end{pmatrix} \begin{pmatrix} \psi^{a'}_A \\ \psi^a_{\dot A} \end{pmatrix} - \frac{i}{2}\theta^2 \begin{pmatrix} F^a_A \\ F^{a'}_{\dot A} \end{pmatrix}. \tag{4.1}$$

The index $A$ of the LHS runs from 1 to 4, while $A$ and $\dot A$ of the RHS run from 1 to 2. (For the dotted and undotted representation, see Appendix A.3.) The indices $a$ and $a'$ run from 1 to $2M$ and 1 to $2N$, respectively. The superfields $\Phi^a_{\dot A}$ and $\Phi^{a'}_{\dot A}$ are called untwisted and twisted hypermultiplets, respectively, in the literature [35] (from the $\mathcal{N}=4$ point of view). The two antisymmetric matrices $\omega^{IJ}$ and $\omega^{AB}$ are decomposed as

$$\omega^{IJ} = \begin{pmatrix} \omega^{ab} & 0 \\ 0 & \omega^{a'b'} \end{pmatrix} \quad \text{and} \quad \omega^{AB} = \begin{pmatrix} \epsilon^{AB} & 0 \\ 0 & \epsilon^{\dot A\dot B} \end{pmatrix} \tag{4.2}$$

respectively. Now the reality condition $(\bar\Phi^A_I)_{\mathcal{N}=5} = \omega^{AB}\omega_{IJ}\Phi^J_B$ becomes

$$\bar\Phi^A_a = \epsilon^{AB}\omega_{ab}\Phi^b_B \quad \text{and} \quad \bar\Phi^{\dot A}_{a'} = \epsilon^{\dot A\dot B}\omega_{a'b'}\Phi^{b'}_{\dot B}. \tag{4.3}$$

To be compatible with the decomposition of the $\mathcal{N}=5$ hypermultiplets (4.1), one may decompose the $\mathcal{N}=5$ superconnections as

$$\Gamma^{IJ} f_{IJ}{}^K{}_L = \begin{pmatrix} \Gamma^{ab} f_{ab}{}^c{}_d + \Gamma^{a'b'} f_{a'b'}{}^c{}_d & 0 \\ 0 & \Gamma^{a'b'} f_{a'b'}{}^{c'}{}_{d'} + \Gamma^{ab} f_{ab}{}^{c'}{}_{d'} \end{pmatrix}, \tag{4.4}$$

where

$$\Gamma^{ab} f_{ab}{}^c{}_d = (i\theta^\beta A^{ab}_{\alpha\beta} + \theta^2 \chi^{ab}_\alpha) f_{ab}{}^c{}_d, \tag{4.5}$$

and the other 3 superfields of the RHS of (4.4) have similar expressions. In proposing (4.4), we have decomposed the set of 3-algebra generators $T_I$ into two sets of generators

$T_a$ and $T_{a'}$, and decomposed the 3-bracket (2.1) into 4 sets, with the structure constants ${f_{abc}}^d$, ${f_{abc'}}^{d'}$, ${f_{a'b'c}}^d$ and ${f_{a'b'c'}}^{d'}$. We have also decomposed the parameter superfield $\Gamma^{IJ}$ into two superfields $\Gamma^{ab}$ and $\Gamma^{a'b'}$.

If we introduce a 'spin up' spinor $\chi_{1\alpha}$ and a 'spin down' spinor $\chi_{2\alpha}$, i.e., [1]

$$\chi_{1\alpha} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \delta_{1\alpha} \quad \text{and} \quad \chi_{2\alpha} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \delta_{2\alpha}, \tag{4.6}$$

then in component formalism, we now have

$$f_{IJKL} = f_{abcd}\delta_{1\alpha}\delta_{1\beta}\delta_{1\gamma}\delta_{1\delta} + f_{abc'd'}\delta_{1\alpha}\delta_{1\beta}\delta_{2\gamma}\delta_{2\delta} + f_{a'b'cd}\delta_{2\alpha}\delta_{2\beta}\delta_{1\gamma}\delta_{1\delta} + f_{a'b'c'd'}\delta_{2\alpha}\delta_{2\beta}\delta_{2\gamma}\delta_{2\delta}, \tag{4.7}$$

(Here we assume that $(f_{abcd} - f_{abc'd'})$ does not vanish identically.) and

$$\Gamma^{IJ} = \Gamma^{ab}\delta_{1\alpha}\delta_{1\beta} + \Gamma^{a'b'}\delta_{2\alpha}\delta_{2\beta}. \tag{4.8}$$

Substituting (4.7) and (4.8) into $\Gamma^{IJ}{f_{IJ}}^K{}_L$ indeed gives (4.4). With the decomposition (4.7), the FI (2.6) are decomposed into 4 sets:

$$\begin{aligned}
&{f_{abe}}^g f_{gfcd} + {f_{abf}}^g f_{egcd} - {f_{efd}}^g f_{abcg} - {f_{efc}}^g f_{abdg} = 0, \\
&{f_{abe}}^g f_{gfc'd'} + {f_{abf}}^g f_{egc'd'} - {f_{efd'}}^{g'} f_{abc'g'} - {f_{efc'}}^{g'} f_{abd'g'} = 0, \\
&{f_{a'b'e}}^g f_{gfc'd'} + {f_{a'b'f}}^g f_{egc'd'} - {f_{efd'}}^{g'} f_{a'b'c'g'} - {f_{efc'}}^{g'} f_{a'b'd'g'} = 0, \\
&{f_{a'b'e'}}^{g'} f_{g'f'c'd'} + {f_{a'b'f'}}^{g'} f_{e'g'c'd'} - {f_{e'f'd'}}^{g'} f_{a'b'c'g'} - {f_{e'f'c'}}^{g'} f_{a'b'd'g'} = 0.
\end{aligned} \tag{4.9}$$

In accordance with Eq. (2.18), these structure constants enjoy the symmetry properties

$$\begin{aligned}
&f_{abcd} = f_{bacd} = f_{badc} = f_{cdab}, \\
&f_{abc'd'} = f_{bac'd'} = f_{bad'c'} = f_{c'd'ab}, \\
&f_{a'b'c'd'} = f_{b'a'c'd'} = f_{b'a'd'c'} = f_{c'd'a'b'}.
\end{aligned} \tag{4.10}$$

The reality condition (2.15) is decomposed into

$$f^{*a}{}_b{}^c{}_d = f^b{}_a{}^d{}_c, \quad f^{*a'}{}_{b'}{}^c{}_d = f^{b'}{}_{a'}{}^d{}_c, \quad f^{*a'}{}_{b'}{}^{c'}{}_{d'} = f^{b'}{}_{a'}{}^{d'}{}_{c'}. \tag{4.11}$$

Under the condition that $(f_{abcd} - f_{abc'd'})$ does not vanish identically, decomposing the constraint condition $f_{(IJK)L} = 0$ results in $f_{(abc)d} = 0$, $f_{(a'b'c')d'} = 0$ and $f_{abc'd'} = 0$. However, the condition $f_{abc'd'} = 0$ turns out to be too restrictive to allow any interaction

[1] Here the index $\alpha$ is *not* an index of a spacetime spinor. We hope this will not cause any confusion.

between the primed fields and the unprimed fields. So we have to give up the constraint $f_{abc'd'} = 0$. Namely, we have to give up the constraint condition $f_{(IJK)L} = 0$ as we decompose $f_{IJKL}$ by Eq. (4.7). Later we will see, to construct an interesting $\mathcal{N} = 4$ quiver gauge theory, we need only to impose constraints on $f_{abcd}$ and $f_{a'b'c'd'}$:

$$f_{(abc)d} = 0 \quad \text{and} \quad f_{(a'b'c')d'} = 0, \tag{4.12}$$

while $f_{abc'd'}$ are unconstrained.

With these decompositions, the Lagrangian for the kinetic terms of the matter fields (3.16) becomes

$$\begin{aligned}\mathcal{L}_{\text{kin}} &= \frac{1}{2}(-D_\mu \bar{Z}^A_a D^\mu Z^a_A + i\bar{\psi}^{\dot{A}}_a \gamma^\mu D_\mu \psi^a_{\dot{A}} - 2i\sigma^{\dagger}_{\dot{B}}{}^{A}\bar{\psi}^{\dot{B}}_a \tilde{\chi}^a{}_b Z^b_A + \bar{F}^A_a F^a_A) \\ &\quad + \frac{1}{2}(-D_\mu \bar{Z}^{\dot{A}}_{a'} D^\mu Z^{a'}_{\dot{A}} + i\bar{\psi}^A_{a'}\gamma^\mu D_\mu \psi^{a'}_A - 2i\sigma_B{}^{\dot{A}}\bar{\psi}^B_{a'}\tilde{\chi}^{a'}{}_{b'} Z^{b'}_{\dot{A}} + \bar{F}^{\dot{A}}_{a'} F^{a'}_{\dot{A}}),\end{aligned} \tag{4.13}$$

where

$$\begin{aligned} D_\mu Z^A_d &= \partial_\mu Z^A_d - \tilde{A}_\mu{}^c{}_d Z^A_c, \\ \tilde{A}_\mu{}^c{}_d &= A^{ab}_\mu f_{ab}{}^c{}_d + A^{a'b'}_\mu f_{a'b'}{}^c{}_d, \\ \tilde{\chi}^{a'}{}_{b'} &= \chi^{cd} f_{cd}{}^{a'}{}_{b'} + \chi^{c'd'} f_{c'd'}{}^{a'}{}_{b'}, \end{aligned} \tag{4.14}$$

and similar definitions for $\tilde{A}_\mu{}^{c'}{}_{d'}$ and $\tilde{\chi}^a{}_b$; and the Chern-Simons term (3.19) becomes

$$\begin{aligned}\mathcal{L}_{\text{CS}} &= \frac{1}{2}\epsilon^{\mu\nu\lambda}(f_{abcd}A^{ab}_\mu \partial_\nu A^{cd}_\lambda + \frac{2}{3} f_{abc}{}^g f_{gdef} A^{ab}_\mu A^{cd}_\nu A^{ef}_\lambda) \\ &\quad + \frac{1}{2}\epsilon^{\mu\nu\lambda}(f_{a'b'c'd'}A^{a'b'}_\mu \partial_\nu A^{c'd'}_\lambda + \frac{2}{3} f_{a'b'c'}{}^{g'} f_{g'd'e'f'} A^{a'b'}_\mu A^{c'd'}_\nu A^{e'f'}_\lambda) \\ &\quad + \epsilon^{\mu\nu\lambda}(f_{abc'd'}A^{ab}_\mu \partial_\nu A^{c'd'}_\lambda + f_{abc}{}^g f_{gde'f'} A^{ab}_\mu A^{cd}_\nu A^{e'f'}_\lambda + f_{abc'}{}^{g'} f_{g'd'e'f'} A^{ab}_\mu A^{c'd'}_\nu A^{e'f'}_\lambda) \\ &\quad + \frac{i}{2}(f_{abcd}\chi^{ab}\chi^{cd} + 2 f_{abc'd'}\chi^{ab}\chi^{c'd'} + f_{a'b'c'd'}\chi^{a'b'}\chi^{c'd'}).\end{aligned} \tag{4.15}$$

The equations of motion for the auxiliary field $\chi$ (3.20) is decomposed into two sets

$$\begin{aligned}\chi^{ab} &= -\sigma^{\dagger}_{\dot{A}}{}^{B}\psi^{\dot{A}(a} Z^{b)}_B, \\ \chi^{a'b'} &= -\sigma_A{}^{\dot{B}}\psi^{A(a'} Z^{b')}_{\dot{B}}.\end{aligned} \tag{4.16}$$

Plugging (4.16) into (4.13) and (4.15) gives three Yukawa terms

$$\begin{aligned}&-\frac{i}{2}(f_{acbd}\sigma^{A\dot{C}}\sigma^{B\dot{D}} Z^a_A Z^b_B \psi^c_{\dot{C}}\psi^d_{\dot{D}} + f_{a'c'b'd'}\sigma^{\dagger\dot{A}C}\sigma^{\dagger\dot{B}D} Z^{a'}_{\dot{A}} Z^{b'}_{\dot{B}}\psi^{c'}_C\psi^{d'}_D \\ &\qquad + 2 f_{abc'd'}\sigma^{A\dot{B}}\sigma^{\dagger\dot{C}D} Z^a_A Z^{c'}_{\dot{C}}\psi^b_{\dot{B}}\psi^{d'}_D).\end{aligned} \tag{4.17}$$

Alternatively, we can also obtain (4.17) by directly decomposing the $\mathcal{N} = 5$ Yukawa term (3.21). It can be seen that the last term of (4.17) is a mixed term, in which the primed fields couple the unprimed fields through $f_{abc'd'}$. So we cannot obtain a nontrivial $\mathcal{N}=4$ superpotential by decomposing the $\mathcal{N}=5$ superpotential (3.24), because the $\mathcal{N}=5$ superpotential (3.24) is desired only if $f_{(IJK)L}=0$, which implies that $f_{abc'd'}=0$ as we decompose $f_{IJKL}$ by Eq. (4.7) under the condition that $(f_{abcd}-f_{abc'd'})$ does not vanish identically. So we have to propose a new superpotential for the $\mathcal{N}=4$ theory, allowing $f_{abc'd'}\neq 0$. However, unlike the last term of (4.17), the first two terms of (4.17) are un-mixed terms. This inspires us to decompose the first term of the $\mathcal{N}=5$ superpotential (3.23) with $f_{a'c'bd}$ and $f_{acb'd'}$ deleted from $f_{IKJL}$ (hence we denote the 'modified' structure constants as $f'_{IKJL}$):

$$\begin{aligned} W_1(\Phi) &= \frac{1}{12}(f'_{IKJL}\omega^{AB}\omega^{CD}\Phi^I_A\Phi^J_B\Phi^K_C\Phi^L_D)_{\mathcal{N}=5} \\ &= \frac{1}{12}(f_{acbd}\epsilon^{AB}\epsilon^{CD}\Phi^a_A\Phi^b_B\Phi^c_C\Phi^d_D + f_{a'c'b'd'}\epsilon^{\dot A\dot B}\epsilon^{\dot C\dot D}\Phi^{a'}_{\dot A}\Phi^{b'}_{\dot B}\Phi^{c'}_{\dot C}\Phi^{d'}_{\dot D}). \end{aligned} \tag{4.18}$$

where

$$f'_{IJKL} = f_{abcd}\delta_{1\alpha}\delta_{1\beta}\delta_{1\gamma}\delta_{1\delta} + f_{a'b'c'd'}\delta_{2\alpha}\delta_{2\beta}\delta_{2\gamma}\delta_{2\delta}. \tag{4.19}$$

Of course, the 'modified' structure constants $f'_{IKJL}$ still satisfy the constraint condition $f'_{(IKJ)L}$=0, which is equivalent to Eq. (4.12): $f_{(acb)d}=0$ and $f_{(a'c'b')d'}=0$. We will prove that the first two terms of (4.17) combining the Yukawa terms arising from the superpotential $W_1$ (see (4.20)) are $SU(2)\times SU(2)$ invariant. Carrying out the Berezin integral $\frac{i}{2}\int d\theta^2 W_1(\Phi)$ gives:

$$\begin{aligned} \mathcal{L}_{W_1} = &-\frac{i}{6}(f_{acbd}\epsilon^{AB}\epsilon^{\dot C\dot D}Z^a_A Z^b_B\psi^c_{\dot C}\psi^d_{\dot D} + f_{a'c'b'd'}\epsilon^{\dot A\dot B}\epsilon^{CD}Z^{a'}_{\dot A}Z^{b'}_{\dot B}\psi^{c'}_C\psi^{d'}_D) \\ &-\frac{i}{6}[(f_{abcd}-f_{adcb})\sigma^{A\dot C}\sigma^{B\dot D}Z^a_A Z^b_B\psi^c_{\dot C}\psi^d_{\dot D} \\ &+(f_{a'b'c'd'}-f_{a'd'c'b'})\sigma^{\dagger\dot A C}\sigma^{\dagger\dot B D}Z^{a'}_{\dot A}Z^{b'}_{\dot B}\psi^{c'}_C\psi^{d'}_D] \\ &-\frac{1}{3}(f_{abcd}Z^b_B Z^{Bc}Z^{Ad}F^a_A + f_{a'b'c'd'}Z^{b'}_{\dot B}Z^{\dot Bc'}Z^{\dot Ad'}F^{a'}_{\dot A}). \end{aligned} \tag{4.20}$$

Let us now combine the first term of (4.17) and the first term of the second line of (4.20):

$$\begin{aligned} &-\frac{i}{6}[3f_{acbd}+(f_{abcd}-f_{adcb})]\sigma^{A\dot C}\sigma^{B\dot D}Z^a_A Z^b_B\psi^c_{\dot C}\psi^d_{\dot D} \\ =&-\frac{i}{6}(f_{acbd}-f_{bcad})(\sigma^{A\dot C}\sigma^{B\dot D}-\sigma^{B\dot C}\sigma^{A\dot D})Z^a_A Z^b_B\psi^c_{\dot C}\psi^d_{\dot D} \\ =&-\frac{i}{3}f_{acbd}\epsilon^{AB}\epsilon^{\dot C\dot D}Z^a_A Z^b_B\psi^c_{\dot C}\psi^d_{\dot D}. \end{aligned} \tag{4.21}$$

In the second line we have used $f_{(abc)d}=0$. In the third line we have used the $SU(2)\times SU(2)$ identity (A.26). It can be seen that the final expression of (4.21) is indeed $SU(2)\times$

$SU(2)$ invariant. Similarly, one can combine the second term of (4.17) and the second term of the second line of (4.20) to form an $SU(2) \times SU(2)$ invariant expression:

$$-\frac{i}{3} f_{a'c'b'd'} \epsilon^{\dot{A}\dot{B}} \epsilon^{CD} Z^{a'}_{\dot{A}} Z^{b'}_{\dot{B}} \psi^{c'}_{C} \psi^{d'}_{D}, \tag{4.22}$$

where we have used the reality condition (A.23). Now only the last term of (4.17), i.e., the mixed term, is not $SU(2) \times SU(2)$ invariant. Its structure suggests that if a Yukawa term of the form

$$i f_{abc'd'} \sigma^{D\dot{B}} \sigma^{\dagger \dot{C}A} Z^{a}_{A} Z^{c'}_{\dot{C}} \psi^{b}_{\dot{B}} \psi^{d'}_{D} \tag{4.23}$$

arises from a to-be-determined superpotential, then they will add up to be $SU(2) \times SU(2)$ invariant by the reality condition (A.23) and the identity (A.26). It is therefore natural to try

$$W_2(\Phi) = \alpha f_{abc'd'} \sigma^{B\dot{D}} \sigma^{\dagger \dot{C}A} \Phi^{a}_{A} \Phi^{b}_{B} \Phi^{c'}_{\dot{C}} \Phi^{d'}_{\dot{D}}, \tag{4.24}$$

where $\alpha$ is a constant, to be determined later. The corresponding Lagrangian is

$$\begin{aligned} \mathcal{L}_{W_2} \;=\; & i\alpha f_{abc'd'} (\epsilon^{AC} \epsilon^{BD} Z^{a}_{A} Z^{b}_{B} \psi^{c'}_{C} \psi^{d'}_{D} + \epsilon^{\dot{A}\dot{C}} \epsilon^{\dot{B}\dot{D}} \psi^{a}_{\dot{A}} \psi^{b}_{\dot{B}} Z^{c'}_{\dot{C}} Z^{d'}_{\dot{D}} + 2\epsilon^{AC} \epsilon^{\dot{B}\dot{D}} Z^{a}_{A} Z^{d'}_{\dot{D}} \psi^{b}_{\dot{B}} \psi^{c'}_{C}) \\ & + 2i\alpha f_{abc'd'} \sigma^{D\dot{B}} \sigma^{\dagger \dot{C}A} Z^{a}_{A} Z^{c'}_{\dot{C}} \psi^{b}_{\dot{B}} \psi^{d'}_{D} \\ & - 2\alpha f_{abc'd'} \sigma^{B\dot{D}} \sigma^{\dagger \dot{C}A} Z^{b}_{B} Z^{c'}_{\dot{C}} Z^{d'}_{\dot{D}} F^{a}_{A} - 2\alpha f_{abc'd'} \sigma^{B\dot{D}} \sigma^{\dagger \dot{C}A} Z^{a}_{A} Z^{b}_{B} Z^{d'}_{\dot{D}} F^{c'}_{\dot{C}}. \end{aligned} \tag{4.25}$$

Note that the first line is $SU(2) \times SU(2)$ invariant by itself. Comparing the second line with (4.23) gives $\alpha = \frac{1}{2}$. Combining the last term of (4.17) and the second line of (4.25), we obtain

$$i f_{abc'd'} \epsilon^{AD} \epsilon^{\dot{B}\dot{C}} Z^{a}_{A} Z^{c'}_{\dot{C}} \psi^{b}_{\dot{B}} \psi^{d'}_{D}, \tag{4.26}$$

which is the desired result. Now all Yukawa terms are invariant under the $SU(2) \times SU(2)$ global symmetry transformation. Put all Yukawa terms (the first line of (4.20), (4.21), (4.22), (4.26) and the first line of (4.25)) together:

$$\begin{aligned} \mathcal{L}_{\mathrm{Y}} \;=\; & -\frac{i}{2} (f_{acbd} Z^{a}_{A} Z^{Ab} \psi^{c}_{\dot{B}} \psi^{\dot{B}d} + f_{a'c'b'd'} Z^{a'}_{\dot{A}} Z^{\dot{A}b'} \psi^{c'}_{B} \psi^{Bd'}) \\ & + \frac{i}{2} f_{abc'd'} (Z^{a}_{A} Z^{b}_{B} \psi^{Ac'} \psi^{Bd'} + Z^{c'}_{\dot{A}} Z^{d'}_{\dot{B}} \psi^{\dot{A}a} \psi^{\dot{B}b} + 4 Z^{a}_{A} Z^{\dot{B}d'} \psi^{b}_{\dot{B}} \psi^{Ac'}). \end{aligned}$$

To calculate the bosonic potential, we first integrate out the auxiliary fields $F^{a}_{A}$ and $F^{a'}_{\dot{A}}$ from (4.13), (4.20) and (4.25):

$$\begin{aligned} \bar{F}^{A}_{a} &= \frac{1}{3} f_{abcd} Z^{b}_{B} Z^{Bc} Z^{Ad} + f_{abc'd'} \sigma^{B\dot{D}} \sigma^{\dagger \dot{C}A} Z^{b}_{B} Z^{c'}_{\dot{C}} Z^{d'}_{\dot{D}} \equiv W^{A}_{1a} + W^{A}_{2a}, \\ \bar{F}^{\dot{A}}_{a'} &= \frac{1}{3} f_{a'b'c'd'} Z^{b'}_{\dot{B}} Z^{\dot{B}c'} Z^{\dot{A}d'} + f_{a'b'cd} \sigma^{\dagger \dot{B}D} \sigma^{C\dot{A}} Z^{b'}_{\dot{B}} Z^{c}_{C} Z^{d}_{D} \equiv W^{\dot{A}}_{1a'} + W^{\dot{A}}_{2a'}. \end{aligned} \tag{4.27}$$

The bosonic potential is

$$-V = -\frac{1}{2}(\bar{F}^A_a F^a_A + \bar{F}^{\dot{A}}_{a'} F^{a'}_{\dot{A}}), \tag{4.28}$$

which is not manifestly $SU(2)\times SU(2)$ invariant due to the presence of the sigma matrices. However, by using the fundamental identities (4.9) and a method first introduced in GW theory [33] (see also [34]), we are able to rewrite (4.28) so that it has a manifest $SU(2) \times SU(2)$ global symmetry. For example, let us consider

$$\begin{aligned} -W^A_{1a} W^a_{2A} &= -\frac{1}{3} f_{abcd} f^a{}_{ec'd'} \sigma^{A\dot{C}} \sigma^{C\dot{D}} Z^b_B Z^{Bc} Z^d_A Z^e_C Z^{c'}_{\dot{C}} Z^{d'}_{\dot{D}} \\ &= -\frac{1}{3}\{f_{cda(b} f^a{}_{e)c'd'} + f_{cda[b} f^a{}_{e]c'd'}\} \sigma^{A\dot{C}} \sigma^{C\dot{D}} Z^b_B Z^{Bc} Z^d_A Z^e_C Z^{c'}_{\dot{C}} Z^{d'}_{\dot{D}} \\ &\equiv S + A. \end{aligned} \tag{4.29}$$

The antisymmetric part can be written as

$$A = \frac{1}{6} f_{cdab} f^a{}_{ec'd'} \sigma^{A\dot{C}} \sigma^{C\dot{D}} Z^{Bb} Z^e_B Z^c_C Z^d_A Z^{c'}_{\dot{C}} Z^{d'}_{\dot{D}}. \tag{4.30}$$

Applying the constraint condition $f_{(cdb)a} = 0$ to the above potential term, we obtain

$$A = -\frac{1}{3} f_{cdae} f^a{}_{bc'd'} \sigma^{A\dot{C}} \sigma^{C\dot{D}} Z^b_B Z^{Bc} Z^d_A Z^e_C Z^{c'}_{\dot{C}} Z^{d'}_{\dot{D}}. \tag{4.31}$$

Combining this with $-W^A_{1a} W^a_{2A}$ (the first line of (4.29)) gives

$$-W^A_{1a} W^a_{2A} + A = 2S. \tag{4.32}$$

Solving for $-W^A_{1a} W^a_{2A}$, we obtain

$$-W^A_{1a} W^a_{2A} = -\frac{1}{2} f_{cda(b} f^a{}_{e)c'd'} \sigma^{A\dot{C}} \sigma^{C\dot{D}} Z^b_B Z^{Bc} Z^d_A Z^e_C Z^{c'}_{\dot{C}} Z^{d'}_{\dot{D}}. \tag{4.33}$$

Let us now consider another term of (4.28):

$$\begin{aligned} -\frac{1}{2} W^{\dot{A}}_{2a'} W^{a'}_{2\dot{A}} &= \frac{1}{2} f_{cdb'a'} f^{a'}{}_{e'fg} \sigma^{D\dot{B}} \sigma^{A\dot{F}} Z^{b'}_{\dot{B}} Z^{e'}_{\dot{F}} Z^c_C Z^d_D Z^{Cf} Z^g_A \qquad (4.34) \\ &= \frac{1}{2}(f_{cda'(b'} f^{a'}{}_{e')fg} + f_{cda'[b'} f^{a'}{}_{e']fg}) \sigma^{D\dot{B}} \sigma^{A\dot{F}} Z^{b'}_{\dot{B}} Z^{e'}_{\dot{F}} Z^c_C Z^d_D Z^{Cf} Z^g_A. \end{aligned}$$

Combining this equation with (4.33), the symmetric part cancels (4.33) by the second equation of the fundamental identities (4.9), while the antisymmetric part is $SU(2) \times SU(2)$ invariant by the identity (A.26). The final result is

$$-W^A_{1a} W^a_{2A} - \frac{1}{2} W^{\dot{A}}_{2a'} W^{a'}_{2\dot{A}} = -\frac{1}{4} f_{abc'g'} f^{g'}{}_{d'ef} Z^{\dot{A}c'} Z^{d'}_{\dot{A}} Z^b_D Z^{Df} Z^a_C Z^{Ce}. \tag{4.35}$$

One can apply the same method to the other terms of (4.28). The final expression for the $\mathcal{N}=4$ bosonic potential is

$$\begin{aligned} -V \;=\;& +\frac{1}{12}(f_{abcg}f^{g}{}_{def}Z^{Aa}Z^{b}_{B}Z^{B(c}Z^{d)}_{C}Z^{Ce}Z^{f}_{A}+f_{a'b'c'g'}f^{g'}{}_{d'e'f'}Z^{\dot{A}a'}Z^{b'}_{\dot{B}}Z^{\dot{B}(c'}Z^{d')}_{\dot{C}}Z^{\dot{C}e'}Z^{f'}_{\dot{A}}) \\ & -\frac{1}{4}(f_{abc'g'}f^{g'}{}_{d'ef}Z^{\dot{A}c'}Z^{d'}_{\dot{A}}Z^{b}_{D}Z^{Df}Z^{a}_{C}Z^{Ce}+f_{a'b'cg}f^{g}{}_{de'f'}Z^{Ac}Z^{d}_{A}Z^{b'}_{\dot{D}}Z^{\dot{D}f'}Z^{a'}_{\dot{C}}Z^{\dot{C}e'}) \end{aligned} \tag{4.36}$$

In summary, the full $\mathcal{N}=4$ Lagrangian is given by

$$\begin{aligned} \mathcal{L} \;=\;& \frac{1}{2}(-D_\mu\bar{Z}^{A}_{a}D^\mu Z^{a}_{A}-D_\mu\bar{Z}^{\dot{A}}_{a'}D^\mu Z^{a'}_{\dot{A}}+i\bar{\psi}^{\dot{A}}_{a}\gamma^\mu D_\mu\psi^{a}_{\dot{A}}+i\bar{\psi}^{A}_{a'}\gamma^\mu D_\mu\psi^{a'}_{A}) \\ & -\frac{i}{2}(f_{acbd}Z^{a}_{A}Z^{Ab}\psi^{c}_{\dot{B}}\psi^{\dot{B}d}+f_{a'c'b'd'}Z^{a'}_{\dot{A}}Z^{\dot{A}b'}\psi^{c'}_{B}\psi^{Bd'}) \\ & +\frac{i}{2}f_{abc'd'}(Z^{a}_{A}Z^{b}_{B}\psi^{Ac'}\psi^{Bd'}+Z^{c'}_{\dot{A}}Z^{d'}_{\dot{B}}\psi^{\dot{A}a}\psi^{\dot{B}b}+4Z^{a}_{A}Z^{\dot{B}d'}\psi^{b}_{\dot{B}}\psi^{Ac'}) \\ & +\frac{1}{2}\epsilon^{\mu\nu\lambda}(f_{abcd}A^{ab}_{\mu}\partial_\nu A^{cd}_{\lambda}+\frac{2}{3}f_{abc}{}^{g}f_{gdef}A^{ab}_{\mu}A^{cd}_{\nu}A^{ef}_{\lambda}) \\ & +\frac{1}{2}\epsilon^{\mu\nu\lambda}(f_{a'b'c'd'}A^{a'b'}_{\mu}\partial_\nu A^{c'd'}_{\lambda}+\frac{2}{3}f_{a'b'c'}{}^{g'}f_{g'd'e'f'}A^{a'b'}_{\mu}A^{c'd'}_{\nu}A^{e'f'}_{\lambda}) \\ & +\epsilon^{\mu\nu\lambda}(f_{abc'd'}A^{ab}_{\mu}\partial_\nu A^{c'd'}_{\lambda}+f_{abc}{}^{g}f_{gde'f'}A^{ab}_{\mu}A^{cd}_{\nu}A^{e'f'}_{\lambda}+f_{abc'}{}^{g'}f_{g'd'e'f'}A^{ab}_{\mu}A^{c'd'}_{\nu}A^{e'f'}_{\lambda}) \\ & +\frac{1}{12}(f_{abcg}f^{g}{}_{def}Z^{Aa}Z^{b}_{B}Z^{B(c}Z^{d)}_{C}Z^{Ce}Z^{f}_{A}+f_{a'b'c'g'}f^{g'}{}_{d'e'f'}Z^{\dot{A}a'}Z^{b'}_{\dot{B}}Z^{\dot{B}(c'}Z^{d')}_{\dot{C}}Z^{\dot{C}e'}Z^{f'}_{\dot{A}}) \\ & -\frac{1}{4}(f_{abc'g'}f^{g'}{}_{d'ef}Z^{\dot{A}c'}Z^{d'}_{\dot{A}}Z^{b}_{D}Z^{Df}Z^{a}_{C}Z^{Ce}+f_{a'b'cg}f^{g}{}_{de'f'}Z^{Ac}Z^{d}_{A}Z^{b'}_{\dot{D}}Z^{\dot{D}f'}Z^{a'}_{\dot{C}}Z^{\dot{C}e'}). \end{aligned} \tag{4.37}$$

Using the same argument given in section 3.1, we may promote the $\mathcal{N}=1$ supersymmetry transformations to $\mathcal{N}=4$:

$$\begin{aligned} &\delta Z^{a}_{A} = i\epsilon_{A}{}^{\dot{A}}\psi^{a}_{\dot{A}}, \\ &\delta Z^{a'}_{\dot{A}} = i\epsilon^{\dagger}_{\dot{A}}{}^{A}\psi^{a'}_{A}, \\ &\delta\psi^{a'}_{A} = -\gamma^\mu D_\mu Z^{a'}_{\dot{B}}\epsilon_{A}{}^{\dot{B}}-\frac{1}{3}f^{a'}{}_{b'c'd'}Z^{b'}_{\dot{B}}Z^{\dot{B}c'}Z^{d'}_{\dot{C}}\epsilon_{A}{}^{\dot{C}}+f^{a'}{}_{b'cd}Z^{b'}_{\dot{A}}Z^{Bc}Z^{d}_{A}\epsilon_{B}{}^{\dot{A}}, \\ &\delta\psi^{a}_{\dot{A}} = -\gamma^\mu D_\mu Z^{a}_{B}\epsilon^{\dagger}_{\dot{A}}{}^{B}-\frac{1}{3}f^{a}{}_{bcd}Z^{b}_{B}Z^{Bc}Z^{d}_{C}\epsilon^{\dagger}_{\dot{A}}{}^{C}+f^{a}{}_{bc'd'}Z^{b}_{A}Z^{\dot{B}c'}Z^{d'}_{\dot{A}}\epsilon^{\dagger}_{\dot{B}}{}^{A}, \\ &\delta\tilde{A}_{\mu}{}^{c}{}_{d} = i\epsilon^{A\dot{B}}\gamma_\mu\psi^{b}_{\dot{B}}Z^{a}_{A}f_{ab}{}^{c}{}_{d}+i\epsilon^{\dagger\dot{A}B}\gamma_\mu\psi^{b'}_{B}Z^{a'}_{\dot{A}}f_{a'b'}{}^{c}{}_{d}, \\ &\delta\tilde{A}_{\mu}{}^{c'}{}_{d'} = i\epsilon^{A\dot{B}}\gamma_\mu\psi^{b}_{\dot{B}}Z^{a}_{A}f_{ab}{}^{c'}{}_{d'}+i\epsilon^{\dagger\dot{A}B}\gamma_\mu\psi^{b'}_{B}Z^{a'}_{\dot{A}}f_{a'b'}{}^{c'}{}_{d'}, \end{aligned} \tag{4.38}$$

where the parameter satisfies the reality condition

$$\epsilon^{\dagger}{}_{\dot{A}}{}^{B} = -\epsilon^{BC}\epsilon_{\dot{A}\dot{B}}\epsilon_{C}{}^{\dot{B}}. \tag{4.39}$$

It is still necessary to verify the closure of the $\mathcal{N}=4$ superalgebra; this will be done in the next subsection. The ordinary Lie algebra counterparts of the Lagrangian (4.37) and the supersymmetry transformations (4.38) are first constructed in Ref. [34]. If $f_{abc'd'} = f_{abcd}$, then $f_{abc'd'}$ also satisfy the constraint equation, i.e., $f_{(abc')d'} = 0$. In this special case, the $\mathcal{N} = 4$ supersymmetry will be enhanced to $\mathcal{N} = 5$.

If one sets the 'twisted' multiplet to be zero, i.e., $\Phi^{a'}_{\dot{A}} = 0$, then (4.37) and (4.38) become the the Lagrangian and the supersymmetry law of the GW theory [33], respectively, in the 3-algebra approach:

$$
\begin{aligned}
\mathcal{L} \;=\;& \frac{1}{2}(-D_\mu \bar{Z}^A_a D^\mu Z^a_A + i\bar{\psi}^{\dot{A}}_a \gamma^\mu D_\mu \psi^a_{\dot{A}}) - \frac{i}{2} f_{acbd} Z^a_A Z^{Ab} \psi^c_{\dot{B}} \psi^{\dot{B}d} \\
&+\frac{1}{2}\epsilon^{\mu\nu\lambda}(f_{abcd} A^{ab}_\mu \partial_\nu A^{cd}_\lambda + \frac{2}{3} f_{abc}{}^g f_{gdef} A^{ab}_\mu A^{cd}_\nu A^{ef}_\lambda) \\
&+\frac{1}{12} f_{abcg} f^g{}_{def} Z^{Aa} Z^b_B Z^{B(c} Z^{d)}_C Z^{Ce} Z^f_A,
\end{aligned}
\tag{4.40}
$$

and

$$
\begin{aligned}
&\delta Z^a_A = i\epsilon_A{}^{\dot{A}} \psi^a_{\dot{A}}, \\
&\delta \psi^a_{\dot{A}} = -\gamma^\mu D_\mu Z^a_B \epsilon^{\dagger}_{\dot{A}}{}^B - \frac{1}{3} f^a{}_{bcd} Z^b_B Z^{Bc} Z^d_C \epsilon^{\dagger}_{\dot{A}}{}^C, \\
&\delta \tilde{A}_\mu{}^c{}_d = i\epsilon^{A\dot{B}} \gamma_\mu \psi^b_{\dot{B}} Z^a_A f_{ab}{}^c{}_d.
\end{aligned}
\tag{4.41}
$$

## 4.2 Closure of the $\mathcal{N}$=4 Algebra

The closure of the algebra of the GW theory was checked in [33]. To our knowledge, there is no explicit check in the literature for the closure of the $\mathcal{N} = 4$ algebra after adding the twisted multiplets into the GW theory. Here we present such a check by starting with the supersymmetry transformation of the scalar fields:

$$
\begin{aligned}
[\delta_1, \delta_2] Z^a_A \;=\;& v^\mu D_\mu Z^a_A + \frac{1}{3} f^a{}_{bcd} Z^b_B Z^c_C Z^d_D \epsilon_{AE} \epsilon^{BC} u^{ED} \\
&+ i f^a{}_{bc'd'} Z^{Bb} Z^{\dot{B}c'} Z^{\dot{A}d'} (\epsilon_{2A\dot{A}} \epsilon^\dagger_{1\dot{B}B} - \epsilon_{1A\dot{A}} \epsilon^\dagger_{2\dot{B}B}),
\end{aligned}
\tag{4.42}
$$

where

$$
v^\mu \equiv i\epsilon^{\dagger}_{1\dot{A}}{}^B \gamma^\mu \epsilon_{2B}{}^{\dot{A}}, \qquad u^{ED} \equiv i(\epsilon^{E\dot{A}}_1 \epsilon^{\dagger}_{2\dot{A}}{}^D - \epsilon^{E\dot{A}}_2 \epsilon^{\dagger}_{1\dot{A}}{}^D). \tag{4.43}
$$

By using the identity $\epsilon_{AE}\epsilon^{BC} = -(\delta_A{}^B \delta_E{}^C - \delta_E{}^B \delta_A{}^C)$, the second term of the RHS of (4.42) can be written as

$$
-\frac{1}{3} f^a{}_{bcd} Z^b_A Z^c_C Z^d_D u^{CD} + \frac{1}{3} f^a{}_{bcd} Z^b_B Z^c_A Z^d_D u^{BD}. \tag{4.44}
$$

The second term is equal to the first term minus the second term by the constraint condition $f^a{}_{(bcd)} = 0$:

$$\frac{1}{3} f^a{}_{bcd} Z^b_B Z^c_A Z^d_D u^{BD} = -\frac{1}{3} f^a{}_{bcd} Z^b_A Z^c_C Z^d_D u^{CD} - \frac{1}{3} f^a{}_{bcd} Z^b_B Z^c_A Z^d_D u^{BD}. \quad (4.45)$$

Therefore the second term of the RHS of (4.42) is equal to

$$-\frac{1}{2} f^a{}_{bcd} Z^c_C Z^d_D u^{CD} Z^b_A. \quad (4.46)$$

By using the fourth equation of (A.27), the second line of the RHS of (4.42) becomes

$$-\frac{1}{2} f^a{}_{bc'd'} Z^{c'}_{\dot{A}} Z^{d'}_{\dot{B}} u^{\dot{A}\dot{B}} Z^b_A, \quad (4.47)$$

where

$$u^{\dot{A}\dot{B}} \equiv i(\epsilon_1^{\dagger \dot{A}C} \epsilon_{2C}{}^{\dot{B}} - \epsilon_2^{\dagger \dot{A}C} \epsilon_{1C}{}^{\dot{B}}). \quad (4.48)$$

In summary, we have

$$[\delta_1, \delta_2] Z^a_A = v^\mu D_\mu Z^a_A + \tilde{\Lambda}^a{}_b Z^b_A. \quad (4.49)$$

While the first is the familiar covariant derivative, the second term is a gauge transformation by a parameter

$$\tilde{\Lambda}^a{}_b \equiv -\frac{1}{2} f^a{}_{bcd} Z^c_C Z^d_D u^{CD} - \frac{1}{2} f^a{}_{bc'd'} Z^{c'}_{\dot{A}} Z^{d'}_{\dot{B}} u^{\dot{A}\dot{B}}. \quad (4.50)$$

Similarly, we have

$$[\delta_1, \delta_2] Z^{a'}_{\dot{A}} = v^\mu D_\mu Z^{a'}_{\dot{A}} + \tilde{\Lambda}^{a'}{}_{b'} Z^{b'}_{\dot{A}}, \quad (4.51)$$

where the parameter $\tilde{\Lambda}^{a'}{}_{b'}$ is defined as

$$\tilde{\Lambda}^{a'}{}_{b'} \equiv -\frac{1}{2} f^{a'}{}_{b'c'd'} Z^{c'}_{\dot{C}} Z^{d'}_{\dot{D}} u^{\dot{C}\dot{D}} - \frac{1}{2} f^{a'}{}_{b'cd} Z^c_A Z^d_B u^{AB}. \quad (4.52)$$

Let us now examine the supersymmetry transformation of the gauge fields:

$$\begin{aligned} [\delta_1, \delta_2] \tilde{A}_\mu{}^a{}_b &= v^\nu \tilde{F}_{\nu\mu}{}^a{}_b - D_\mu \tilde{\Lambda}^a{}_b \\ &\quad + v^\nu \{ \tilde{F}_{\mu\nu}{}^a{}_b - \varepsilon_{\mu\nu\lambda} [(Z^c_A D^\lambda \bar{Z}^{Ad} - \frac{i}{2} \bar{\psi}^{\dot{B}c} \gamma^\lambda \psi^d_{\dot{B}}) f_{cd}{}^a{}_b \\ &\quad + (Z^{c'}_{\dot{A}} D^\lambda \bar{Z}^{\dot{A}d'} - \frac{i}{2} \bar{\psi}^{Bc'} \gamma^\lambda \psi^{d'}_B) f_{c'd'}{}^a{}_b] \} \\ &\quad + \mathcal{O}(Z^4). \end{aligned} \quad (4.53)$$

The last term $\mathcal{O}(Z^4)$, which is fourth order in the scalar fields $Z$, vanishes by the FI (4.9). The second line and the third line must be the equations of motion for the gauge fields:

$$\tilde{F}_{\mu\nu}{}^{a}{}_{b} = \varepsilon_{\mu\nu\lambda}[(Z^{c}_{A} D^{\lambda} \bar{Z}^{Ad} - \frac{i}{2}\bar{\psi}^{\dot{B}c}\gamma^{\lambda}\psi^{d}_{\dot{B}}) f_{cd}{}^{a}{}_{b} + (Z^{c'}_{\dot{A}} D^{\lambda} \bar{Z}^{\dot{A}d'} - \frac{i}{2}\bar{\psi}^{Bc'}\gamma^{\lambda}\psi^{d'}_{B}) f_{c'd'}{}^{a}{}_{b}], \quad (4.54)$$

while the first line remains:

$$[\delta_1, \delta_2]\tilde{A}_{\mu}{}^{a}{}_{b} \quad = \quad v^{\nu}\tilde{F}_{\nu\mu}{}^{a}{}_{b} - D_{\mu}\tilde{\Lambda}^{a}{}_{b}. \quad (4.55)$$

The first term is a covariant translation; the second term is a gauge transformation, as expected. Similarly, we have

$$[\delta_1, \delta_2]\tilde{A}_{\mu}{}^{a'}{}_{b'} \quad = \quad v^{\nu}\tilde{F}_{\nu\mu}{}^{a'}{}_{b'} - D_{\mu}\tilde{\Lambda}^{a'}{}_{b'}, \quad (4.56)$$

and

$$\tilde{F}_{\mu\nu}{}^{a'}{}_{b'} = \varepsilon_{\mu\nu\lambda}[(Z^{c'}_{\dot{A}} D^{\lambda} \bar{Z}^{\dot{A}d'} - \frac{i}{2}\bar{\psi}^{Bc'}\gamma^{\lambda}\psi^{d'}_{B}) f_{c'd'}{}^{a'}{}_{b'} + (Z^{c}_{A} D^{\lambda} \bar{Z}^{Ad} - \frac{i}{2}\bar{\psi}^{\dot{B}c}\gamma^{\lambda}\psi^{d}_{\dot{B}}) f_{cd}{}^{a'}{}_{b'}]. \quad (4.57)$$

Finally we examine the fermion supersymmetry transformation:

$$\begin{aligned}[\delta_1, \delta_2]\psi^{a}_{\dot{A}} \quad = \quad & v^{\mu} D_{\mu}\psi^{a}_{\dot{A}} + \tilde{\Lambda}^{a}{}_{b}\psi^{b}_{\dot{A}} \\ & - \frac{i}{2}(\epsilon_1^{\dagger\dot{C}B}\epsilon_{2B\dot{A}} - \epsilon_2^{\dagger\dot{C}B}\epsilon_{1B\dot{A}}) E^{a}_{\dot{C}} \\ & - \frac{1}{2} v_{\nu}\gamma^{\nu} E^{a}_{\dot{A}}, \end{aligned} \quad (4.58)$$

where

$$E^{a}_{\dot{A}} = \gamma^{\mu} D_{\mu}\psi^{a}_{\dot{A}} + f_{cdb}{}^{a} Z^{b}_{B} Z^{Bc}\psi^{d}_{\dot{A}} - f_{c'd'b}{}^{a} Z^{c'}_{\dot{A}} Z^{d'}_{\dot{C}}\psi^{\dot{C}b} + 2 f_{c'd'b}{}^{a} Z^{b}_{B} Z^{c'}_{\dot{A}}\psi^{Bd'}. \quad (4.59)$$

In order to achieve the closure of the algebra, we must impose the equations of motion for the fermionic fields:

$$E^{a}_{\dot{A}} = 0. \quad (4.60)$$

As a result, only the first line of (4.58) remains. Similarly, we obtain

$$[\delta_1, \delta_2]\psi^{a'}_{A} \quad = \quad v^{\mu} D_{\mu}\psi^{a'}_{A} + \tilde{\Lambda}^{a'}{}_{b'}\psi^{b'}_{A},$$

and

$$0 = E^{a'}_{A} = \gamma^{\mu} D_{\mu}\psi^{a'}_{A} + f_{c'd'b'}{}^{a'} Z^{b'}_{\dot{B}} Z^{\dot{B}c'}\psi^{d'}_{A} - f_{cdb'}{}^{a'} Z^{c}_{A} Z^{d}_{C}\psi^{Cb'} + 2 f_{cdb'}{}^{a'} Z^{b'}_{\dot{B}} Z^{c}_{A}\psi^{\dot{B}d}. \quad (4.61)$$

One can derive all the equations of motion as the Euler-Lagrangian equations from the Lagrangian (4.37).

# CHAPTER 5

# 3-ALGEBRAS, LIE SUPERALGEBRAS AND EMBEDDING TENSORS

## 5.1 3-algebras and Lie Superalgebras

In this section, we will demonstrate that the symplectic 3-algebra can be realized in terms of a super Lie algebra [44].

Recall that in section 3.1, we note that $f_{IJKL}$ can be specified as $k_{mn}\tau^m_{IJ}\tau^n_{KL}$ (up to an unimportant constant), i.e.,

$$f_{IJKL} = k_{mn}\tau^m_{IJ}\tau^n_{KL}, \tag{5.1}$$

where the set of matrices $\tau^m_{IK}$ is in the fundamental representation of $Sp(2L)$ or its subalgebra, and $k_{mn}$ is the Killing-Cartan metric. Later we will prove that (5.1) is an explicit solution of the FI (2.6) (see section 5.2).

Further more, the constraint condition $f_{(IJK)L} = 0$ implies that $f_{(IJK)L} = k_{mn}\tau^m_{(IJ}\tau^n_{K)L} = 0$. As GW pointed out [33], the constraint equation $k_{mn}\tau^m_{(IJ}\tau^n_{K)L} = 0$ can be solved in terms of the Jacobi identity for following super Lie algebra: [1]

$$\begin{aligned}
&[M^m, M^n] = C^{mn}{}_s M^s, \\
&[M^m, Q_I] = -\tau^m_{IJ}\omega^{JK}Q_K, \\
&\{Q_I, Q_J\} = \tau^m_{IJ}k_{mn}M^n.
\end{aligned} \tag{5.2}$$

Namely, the $QQQ$ Jacobi identity

$$[\{Q_I, Q_J\}, Q_K] + [\{Q_J, Q_K\}, Q_I] + [\{Q_K, Q_I\}, Q_J] = 0 \tag{5.3}$$

is equivalent to the constraint equation $k_{mn}\tau^m_{(IJ}\tau^n_{K)L} = 0$. Therefore GW's approach suggests that the symplectic 3-algebra can be realized in terms of the super Lie algebra

[1] This is *not* the $D = 3$ super-Pioncare algebra.

(5.2), if we think of the 3-algebra generator $T_I$ as the fermionic generator $Q_I$. Comparing the 3-bracket $[T_I, T_J; T_K] = f_{IJK}{}^L T_l$ with

$$[\{Q_I, Q_J\}, Q_K] = k_{mn}\tau^m_{IJ}\tau^{n}_{K}{}^{L} Q_L, \tag{5.4}$$

and taking account of (5.1), we see that the 3-bracket may be realized in terms of the double graded commutator

$$[T_I, T_J; T_K] \doteq [\{Q_I, Q_J\}, Q_K]. \tag{5.5}$$

Here the RHS is also obviously symmetric in $IJ$. It is instructive to examine the FI (2.5) with the 3-brackets replaced by the double graded commutators:

$$\begin{aligned} & [\{Q_I, Q_J\}, [\{Q_M, Q_N\}, Q_K]] \\ = \; & [\{[\{Q_I, Q_J\}, Q_M], Q_N\}, Q_K] + [\{Q_M, [\{Q_I, Q_J\}, Q_N]\}, Q_K] \\ & +[\{Q_M, Q_N\}, [\{Q_I, Q_J\}, Q_K]]. \end{aligned} \tag{5.6}$$

By using the super Lie algebra (5.2), we obtain

$$\tau^m_{IJ}\tau^n_{MN}([M_n, [M_m, Q_K]] - [M_m, [M_n, Q_K]] + [[M_m, M_n], Q_K]) = 0, \tag{5.7}$$

which is equivalent to the $MMQ$ Jacobi identity of the super Lie algebra (5.2). It is not difficult to prove that $k_{mn}\tau^m_{IJ}\tau^n_{KL}$ also enjoy the symmetry properties (2.18). So indeed the symplectic 3-algebra can be realized in terms of the super Lie algebra. Now recall the component formalism of the basic definition of the global transformation

$$\delta_{\tilde{\Lambda}} X^K = \Lambda^{IJ} f_{IJ}{}^K{}_L X^L. \tag{5.8}$$

Replacing $f_{IJ}{}^K{}_L$ by $k_{mn}\tau^m_{IJ}\tau^{nK}{}_L$ gives

$$\delta_{\tilde{\Lambda}} X^K = \Lambda^{IJ} k_{mn}\tau^m_{IJ}\tau^{nK}{}_L X^L. \tag{5.9}$$

From the ordinary Lie group point of view, this is a transformation with parameters $\Lambda^{IJ}k_{mn}\tau^m_{IJ}$ and generators $\tau^{nK}{}_L$. On the other hand, the second equation of (5.2) indicates that the fermionic generators furnish a representation of the bosonic part of the super Lie algebra (5.2), i.e., the matrix $\tau^m_{IJ}$ is a quaternion representation of $M^m$. Therefore, the gauge group generated by the 3-algebra can be determined as follows: its Lie algebra is just the bosonic part of the super Lie algebra (5.2), which must be $Sp(2L)$

or its subalgebras. The representation of the matter fields is determined by the fermionic generators of the super Lie algebra (5.2).

For a more mathematical approach, see Ref. [37, 42, 46], in which the relations between the 3-algebras and Lie superalgebras are discussed by using Lie algebra representation theories.

## 5.2 Three-algebras and Lie Algebras

It is less obvious that one can also prove that (5.1) is an explicit solution of the FI (2.6) by using the $QQM$ Jacobi identity of the super Lie algebra, which reads

$$[\{Q_I, Q_J\}, M^m] - \{[Q_J, M^m], Q_I\} + \{[M^m, Q_I], Q_J\} = 0. \tag{5.10}$$

After some algebraic steps we obtain

$$\tau^n_{IJ} k_{np}[M^p, M^m] - \tau^{mK}{}_J \tau^n_{KI} k_{np} M^p - \tau^{mK}{}_I \tau^n_{KJ} k_{np} M^p = 0. \tag{5.11}$$

Since the matrix $\tau^m_{IJ}$ is a representation of $M^m$, the above equation implies

$$\tau^n_{IJ} k_{np}[\tau^p, \tau^m]_{MN} - \tau^{mK}{}_J \tau^n_{KI} k_{np} \tau^p_{MN} - \tau^{mK}{}_I \tau^n_{KJ} k_{np} \tau^p_{MN} = 0, \tag{5.12}$$

where

$$[\tau^p, \tau^m]_{MN} = \tau^p_{MO} \tau^{mO}{}_N - \tau^m_{MO} \tau^{pO}{}_N. \tag{5.13}$$

Multiplying both sides by $k_{mq}\tau^q_{KL}$ gives

$$k_{np}\tau^n_{IJ} k_{mq}\tau^q_{KL}[\tau^p, \tau^m]_{MN} - k_{mq}\tau^q_{KL}\tau^{mK}{}_J \tau^n_{KI} k_{np}\tau^p_{MN} - k_{mq}\tau^q_{KL}\tau^{mK}{}_I \tau^n_{KJ} k_{np}\tau^p_{MN} = 0. \tag{5.14}$$

Rearranging the above equation verifies explicitly that (5.1) satisfies the FI (2.6). Application of the commutator

$$[\tau^m, \tau^n]_{IJ} = C^{mn}{}_p \tau^p_{IJ} \tag{5.15}$$

to Eq. (5.14) gives

$$(k_{np}k_{qm}C^{pm}{}_s + k_{qm}k_{sp}C^{pm}{}_n)\tau^n_{IJ}\tau^q_{KL}\tau^s_{MN} = 0. \tag{5.16}$$

Here the equation in the bracket is simply the statement that the structure constants

$$\tilde{C}_{nqs} = k_{np}k_{qm}C^{pm}{}_s \tag{5.17}$$

are totally antisymmetric if the three adjoint indices $nqs$ are on equal footing. Note that $k_{mn}$ is an invariant bilinear form on the bosonic subalgebra, since Eq. (5.16) or (5.17) also implies

$$[k, C^m] = 0. \tag{5.18}$$

Here the matrices $(C^m)^p{}_n = C^{mp}{}_n$ furnish the usual adjoint representation of the bosonic subalgebra. In this way, we see that the FI of the 3-algebra can be converted into two ordinary commutators (5.15) and (5.18) (this is first discovered in the second paper of Ref. [15] with a different approach).

Eq. (5.9) indicates that $f_{IJKL} = k_{mn}\tau^m_{IJ}\tau^n_{KL}$ also furnish a quaternion representation of the bosonic subalgebra. In fact, if we write $f_{IJKL}$ as $(f_{IJ})_{KL}$, then $f_{IJ}$ is a set of matrices, and corresponding matrix elements are $(f_{IJ})_{KL}$. If $\tau^n_{KL}$ furnish a quaternion of representation of $M^n$, then $(f_{IJ})_{KL}$ furnish a quaternion representation of $M_{IJ} = k_{mn}\tau^m_{IJ}M^n$, since the operator $M_{IJ}$ is a linear combination of $M^n$. With this understanding, we are able to rewrite the FI (2.6) as a commutator

$$\begin{aligned}
[f_{IJ}, f_{KL}]_{MN} &= C_{IJ,KL}{}^{OP}(f_{OP})_{MN} \\
&= (f_{IJK}{}^{O}\delta^P_L + f_{IJL}{}^{O}\delta^P_K)(f_{OP})_{MN} \\
&= -[f_{IJ}, f_{MN}]_{KL}.
\end{aligned} \tag{5.19}$$

The third equation says that the quantity $[f_{IJ}, f_{KL}]_{MN}$ are totally antisymmetric in the 3 pairs of indices. Eq. (5.19) is equivalent to Eq. (5.15). Also, the matrices $(f_{IJ})_{KL}$ satisfy the conventional Jacobi identity as a result of the $MMM$ Jacobi identity of the superalgebra of (5.2). We now must check whether $\tilde{C}_{IJ,KL,MN} = k_{MN,OP}C_{IJ,KL}{}^{OP}$ are totally antisymmetric or not. To be consistent with the transformation $M_{IJ} = k_{mn}\tau^m_{IJ}M^n$, we must transform the Killing-Cartan metric $k^{mn}$ as

$$k^{mn} \to k_{IJ,KL} = k_{qm}\tau^q_{IJ}k_{pn}\tau^p_{KL}k^{mn} = k_{mn}\tau^m_{IJ}\tau^n_{KL} = f_{IJKL}. \tag{5.20}$$

Namely the structure constants $f_{IJKL}$ also play a role of the Killing-Cartan metric $k_{IJ,KL}$. So we must use $f_{MNOP}$ to lower the $OP$ indices of $C_{IJ,KL}{}^{OP}$: [2]

$$\begin{aligned}
\tilde{C}_{IJ,KL,MN} &= f_{MNOP}C_{IJ,KL}{}^{OP} \\
&= [f_{MN}, f_{IJ}]_{KL}.
\end{aligned} \tag{5.21}$$

[2] This is a comment by E. Witten, quoted in the second paper of Ref. [15].

By the third equation of (5.19), the structure constants $\tilde{C}_{IJ,KL,MN}$ are indeed totally antisymmetric in the 3 pairs of indices. Therefore Eq. (5.18) now takes the following form

$$[f, C_{IJ}] = 0 \quad \text{or} \quad [f_{MN}, f_{IJ}]_{KL} + [f_{KL}, f_{IJ}]_{MN} = 0, \tag{5.22}$$

which is nothing but the third equation of Eq. (5.19). Namely both Eq. (5.15) and Eq. (5.18) can be written as the third equation of Eq. (5.19), if we express everything in terms of the 3-algebra structure constants $f_{IJKL}$.

Note that we use $k_{mn}$ to lower an adjoint index, while use $\omega_{IJ}$ to lower a fundamental index. If Eq. (5.1) holds, then Eq. (2.8) implies a compatible condition between $k_{mn}$ and $\omega_{IJ}$. Eq. (2.8) is equivalent to $k_{nm}\tau_I^{mK}\omega_{KJ} + k_{nm}\tau_J^{mK}\omega_{IK} = 0$, i.e.,

$$\tilde{\tau}_{nIJ} - k_{nm}\omega_{IK}\tau_J^{mK} = 0, \tag{5.23}$$

where $\tilde{\tau}_{nIJ} \equiv k_{nm}\tau^m_{IJ}$.

## 5.3 Three-algebras and Embedding Tensors

In Ref. [31, 32], the authors derive some extended superconformal gauge theories by taking a conformal limit of $D = 3$ gauged supergravity theories. In their approach, the embedding tensor plays a crucial role. By definition, the embedding tensor $\theta_{mn} = \theta_{nm}$ acts as a projector [32]:

$$D_\mu = \partial_\mu - A_\mu^m \theta_{mn} t^n, \tag{5.24}$$

where $t^n$ is a set of independent generators. The above equation says that $\theta_{mn}$ projects $t^n$ onto another set of generators $\tilde{t}_m = \theta_{mn}t^n$, whose symmetries are gauged. Let us now consider the commutator

$$[\tilde{t}_m, \tilde{t}_n] = \theta_{mp}\theta_{ns}C^{ps}{}_q t^q. \tag{5.25}$$

Since we expect that $[\tilde{t}_m, \tilde{t}_n] = \tilde{C}_{mn}{}^r\tilde{t}_r$, we must set

$$\theta_{mp}\theta_{ns}C^{ps}{}_q = \tilde{C}_{mn}{}^r\theta_{rq}. \tag{5.26}$$

It is necessary to examine the Jacobi identity

$$\begin{aligned}
& [[\tilde{t}_m, \tilde{t}_n], \tilde{t}_p] + [[\tilde{t}_n, \tilde{t}_p], \tilde{t}_m] + [[\tilde{t}_p, \tilde{t}_m], \tilde{t}_n] \\
= \; & (\tilde{C}_{mn}{}^s\tilde{C}_{sp}{}^r + \tilde{C}_{np}{}^s\tilde{C}_{sm}{}^r + \tilde{C}_{pm}{}^s\tilde{C}_{sn}{}^r)\theta_{rq}t^q \\
= \; & (C^{lq}{}_r C^{rs}{}_t + C^{qs}{}_r C^{rl}{}_t + C^{sl}{}_r C^{rq}{}_t)\theta_{ml}\theta_{nq}\theta_{ps}t^t = 0.
\end{aligned} \tag{5.27}$$

In the last line we have used (5.26). The last line is nothing but the Jacobi identity satisfied by ${C^{mn}}_p$. So Eq. (5.27) is indeed the desired result. To construct a physical theory, the embedding tensor is required to be invariant under the transformations which are gauged. Since the embedding tensor $\theta_{mn}$ carries two adjoint indices, we have to set

$$\tilde{C}_{nq}{}^{r}\theta_{rs} + \tilde{C}_{ns}{}^{r}\theta_{qr} = 0. \tag{5.28}$$

Taking account of (5.26), the above equation is equivalent to

$$\theta_{np}\theta_{qm}C^{pm}{}_{s} + \theta_{np}\theta_{sm}C^{pm}{}_{q} = 0. \tag{5.29}$$

This quadratic constraint takes the same form for all extended supergravity theories. We will focus on the $\mathcal{N} = 5$ case. If we represent the adjoint index $m$ as a pair of fundamental indices $IJ$, the embedding tensor becomes $\theta_{IJ,KL}$, satisfying the same symmetry properties as $f_{IJKL}$ do (see (2.18)) [31]. To construct $\mathcal{N} = 5$ supergravity theories, the embedding tensor is required to satisfy the linear constraint:

$$\theta_{(IJ,K)L} = 0, \tag{5.30}$$

and the structure constants in (5.29) are required to be those of $Sp(2L)$ [31]. We observe that if one identifies the embedding tensor $\theta_{mn}$ with the Killing-Cartan metric $k_{mn}$, Eq. (5.29) is precisely the same as Eq. (5.16), which is the FI satisfied by the 3-algebra structure constants $f_{IJKL} = k_{mn}\tau^m_{IJ}\tau^n_{KL}$. Recall that $f_{IJKL}$ also play the role of the Killing-Cartan metric (see section 5.2). So identifying the embedding tensor with the Killing-Cartan metric is equivalent to identifying the embedding tensor with the 3-algebra structure constants. With this identification, Eq. (5.30) is also solved since it is nothing but $f_{(IJK)L} = 0$. We are therefore led to the conclusion that $f_{IJKL}$ also play the role of the embedding tensor. It is straightforward to generalize the discussion of this section to the cases with other values of $\mathcal{N}$.

In summary, if we realize the symplectic 3-algebra in terms of the superalgebra (5.2), we find that $f_{IJKL} = k_{mn}\tau^m_{IJ}\tau^n_{KL}$ play four roles simultaneously:

- $f_{IJKL}$ are the structure constants of the symplectic 3-algebra or the double graded commutator (5.4);
- $f_{IJKL}$ furnish a quaternion representation of the bosonic part of the superalgebra;
- $f_{IJKL}$ play the role of the Killing-Cartan metric;

- $f_{IJKL}$ are the components of the embedding tensor used to construct the $D = 3$ extended supergravity theories.

# CHAPTER 6

# $\mathcal{N}$=4, 5 THEORIES IN TERMS OF THE BOSONIC PARTS OF SUPERALGEBRAS

The $\mathcal{N} = 4, 5$ theories in Chapter 3 and 4 are constructed in terms of 3-algebras. After the discussions of the last section, we are ready to derive their ordinary Lie Algebra constructions by the solution (5.1) [44].

## 6.1 $\mathcal{N} = 5$ Theories in Terms of the Bosonic Parts of Superalgebras

With the solution

$$f_{IJKL} = k_{mn}\tau^m_{IJ}\tau^n_{KL}, \quad [\tau^m, \tau^n]_{IJ} = C^{mn}{}_p\tau^p_{IJ}, \tag{6.1}$$

the gauge field becomes

$$\tilde{A}_\mu{}^K{}_L = A^{IJ}_\mu f_{IJ}{}^K{}_L = A^{IJ}_\mu \tau^m_{IJ} k_{mn}\tau^{nK}{}_L \equiv A^m_\mu k_{mn}\tau^{nK}{}_L. \tag{6.2}$$

Following Ref. [33], we define the 'momentum map' and 'current ' operator as follows

$$\mu^m_{AB} \equiv \tau^m_{IJ} Z^I_A Z^J_B, \quad j^m_{AB} \equiv \tau^m_{IJ} Z^I_A \psi^J_B. \tag{6.3}$$

Here $A = 1, \cdots, 4$ is the fundamental index of the R-symmetry group $Sp(4)$. Substituting the (6.1) and (6.2) into the Lagrangian (3.38) gives

$$\begin{aligned}
\mathcal{L} \;=\; & \frac{1}{2}(-D_\mu \bar{Z}^A_I D^\mu Z^I_A + i\bar{\psi}^A_I \gamma_\mu D^\mu \psi^I_A) - \frac{i}{2}\omega^{AB}\omega^{CD}k_{mn}(j^m_{AC}j^n_{BD} - 2j^m_{AC}j^n_{DB}) \\
& + \frac{1}{2}\epsilon^{\mu\nu\lambda}(k_{mn}A^m_\mu \partial_\nu A^n_\lambda + \frac{1}{3}\tilde{C}_{mnp}A^m_\mu A^n_\nu A^p_\lambda) \\
& + \frac{1}{30}\tilde{C}_{mnp}\mu^{mA}{}_B\mu^{nB}{}_C\mu^{pC}{}_A + \frac{3}{20}k_{mp}k_{ns}(\tau^m\tau^n)_{IJ}Z^{AI}Z^J_A\mu^{pB}{}_C\mu^{sC}{}_B.
\end{aligned} \tag{6.4}$$

Similarly, with the solution (6.1), the supersymmetry transformation law becomes

$$\begin{aligned}
\delta Z^I_A &= i\epsilon_A{}^B\psi^I_B, \\
\delta\psi^I_A &= \gamma^\mu D_\mu Z^I_B \epsilon^B{}_A + \frac{1}{3}k_{mn}\tau^{mI}{}_J\omega^{BC}Z^J_B\mu^n_{CD}\epsilon^D{}_A - \frac{2}{3}k_{mn}\tau^{mI}{}_J\omega^{BD}Z^J_C\mu^n_{DA}\epsilon^C{}_B, \\
\delta A^m_\mu &= i\epsilon^{AB}\gamma_\mu j^m_{AB}.
\end{aligned} \tag{6.5}$$

Here the parameter $\epsilon_A{}^B$ obeys the traceless condition and the reality condition (3.46). The $\mathcal{N}=5$ Lagrangian (6.4) and supersymmetry transformation law (6.5) are in agreement with those given in Ref. [35], which were derived directly in terms of ordinary Lie algebra.

In section (5.1), we have demonstrated that if the structure constants of the 3-algebra are specified as (6.1), then the Lie algebra of the gauge group generated by the 3-algebra is just the bosonic part of the superalgebra (5.2). The following classical super-Lie algebras:

$$U(M|N), \quad OSp(M|2N), \quad OSp(2|2N), \quad F(4), \quad G(3), \quad D(2|1;\alpha), \tag{6.6}$$

(with $\alpha$ a continuous parameter) are of the same form as that of the superalgebra (5.2). Therefore their bosonic parts can be selected to be the Lie algebras of the gauge groups of the $\mathcal{N}=5$ theories. Especially, if we choose the $U(M|N)$ or $OSp(2|2N)$, whose bosonic part is in the two conjugate representations $(R\oplus\bar{R})$, then the supersymmetry will get enhanced to $\mathcal{N}=6$ [35]. In Appendix C.1, we work out the details of the $\mathcal{N}=5, Sp(2N)\times O(M)$ CSM theory.

## 6.2 $\mathcal{N}=4$ Theories in Terms of the Bosonic Parts of Superalgebras

### 6.2.1 $Sp(2N_1)\times SO(N_2)\times Sp(2N_3)$ Example

In order to realize the 3-algebra used to construct the $\mathcal{N}=4$ theories, one needs to 'fuse' two simple super Lie algebras into a single superalgebra, by requiring that the bosonic parts of these two simple superalgebras to share *one* simple factor. The general structure of this 'fused' will be constructed in a forthcoming paper [45]. In this section, we demonstrate how to 'fuse' a pair of simple superalgebras into one superalgebra by presenting an explicit example. Suppose that the untwisted multiplets are in the bifundamental representation of $Sp(2N_1)\times SO(N_2)$ (the bosonic part of $OSp(N_2|2N_1)$), while the twisted multiplets are in the bifundamental representation of $Sp(2N_3)\times SO(N_4)$ (the bosonic part of $OSp(N_4|2N_3)$). Without loss of generality, we assume that $N_2=N_4$

and $N_1 \neq N_3$, i.e., the two superalgebras share one simple factor $\mathfrak{so}(N_2)$. So the gauge group is $Sp(2N_1) \times SO(N_2) \times Sp(2N_3)$. Let's work out the details. The super Lie algebra $OSp(N_2|2N_1)$ reads

$$\begin{aligned}
&[M_{\bar{i}\bar{j}}, M_{\bar{k}\bar{l}}] = \delta_{\bar{j}\bar{k}} M_{\bar{i}\bar{l}} - \delta_{\bar{i}\bar{k}} M_{\bar{j}\bar{l}} + \delta_{\bar{i}\bar{l}} M_{\bar{j}\bar{k}} - \delta_{\bar{j}\bar{l}} M_{\bar{i}\bar{k}}, \\
&[M_{\hat{i}\hat{j}}, M_{\hat{k}\hat{l}}] = \omega_{\hat{j}\hat{k}} M_{\hat{i}\hat{l}} + \omega_{\hat{i}\hat{k}} M_{\hat{j}\hat{l}} + \omega_{\hat{i}\hat{l}} M_{\hat{j}\hat{k}} + \omega_{\hat{j}\hat{l}} M_{\hat{i}\hat{k}}, \\
&[M_{\bar{i}\bar{j}}, Q_{\bar{k}\hat{k}}] = \delta_{\bar{j}\bar{k}} Q_{\bar{i}\hat{k}} - \delta_{\bar{i}\bar{k}} Q_{\bar{j}\hat{k}}, \\
&[M_{\hat{i}\hat{j}}, Q_{\bar{k}\hat{k}}] = \omega_{\hat{j}\hat{k}} Q_{\bar{k}\hat{i}} + \omega_{\hat{i}\hat{k}} Q_{\bar{k}\hat{j}}, \\
&[Q_{\bar{i}\hat{i}}, Q_{\bar{j}\hat{j}}] = k(\omega_{\hat{i}\hat{j}} M_{\bar{i}\bar{j}} + \delta_{\bar{i}\bar{j}} M_{\hat{i}\hat{j}}),
\end{aligned} \tag{6.7}$$

where $\bar{i} = 1, \cdots, N_2$ is an $SO(N_2)$ fundamental index, and $\hat{i} = 1, \cdots, 2N_1$ is an $Sp(2N_1)$ fundamental index, $Q_a = Q_{\bar{i}\hat{i}}$ and $\omega_{ab} = \omega_{\bar{i}\hat{i},\bar{j}\hat{j}} = \delta_{\bar{i}\bar{j}}\omega_{\hat{i}\hat{j}}$. The super Lie algebra $OSp(N_2|2N_3)$ has similar expressions. We denote the fermionic generators of $OSp(N_2|2N_3)$ as $Q_{a'} = Q_{\bar{i}i'}$, where $i' = 1, \cdots, 2N_3$ is an $Sp(2N_3)$ fundamental index.

Since $Q_{\bar{i}i'}$ also carries an $SO(N_2)$ fundamental index, the anticommutator $\{Q_{\bar{i}\hat{i}}, Q_{\bar{j}j'}\}$ cannot vanish. Actually, if $\{Q_{\bar{i}\hat{i}}, Q_{\bar{j}j'}\} = 0$, then the $Q_{\bar{i}\hat{i}}Q_{\bar{j}j'}Q_{\bar{k}k'}$ Jacobi identity implies that $[M_{\bar{j}\bar{k}}, Q_{\bar{i}\hat{i}}] = 0$, which is contradictory with the third equation of (6.7). Namely, if these two superalgebras share one simple factor, then we indeed have $\{Q_{\bar{i}\hat{i}}, Q_{\bar{j}j'}\} \neq 0$, i.e., this anticommutator must equal to a bosonic operator. On the other hand, since $\hat{i}$ and $j'$ are independent indices (recall $N_1 \neq N_3$), it is natural to define

$$\begin{aligned}
&\{Q_{\bar{i}\hat{i}}, Q_{\bar{j}j'}\} = k\delta_{\bar{i}\bar{j}} M_{\hat{i}j'}, \quad [M_{\hat{i}\hat{j}}, Q_{\bar{k}l'}] = [M_{i'j'}, Q_{\bar{k}\hat{l}}] = 0, \\
&[M_{\hat{i}i'}, Q_{\bar{j}j'}] = \omega_{i'j'} Q_{\bar{j}\hat{i}}, \quad [M_{\hat{i}i'}, Q_{\bar{j}\hat{j}}] = \omega_{\hat{i}\hat{j}} Q_{\bar{j}i'}.
\end{aligned} \tag{6.8}$$

It is not difficult (though a little tedious) to verify that *every* Jacobi identity is satisfied. So the five graded commutators in (6.8) must be the correct ones; they are explicit examples of the last five graded commutators of the new superalgebra in Ref. [45] 'fused' by two simple superalgebras.

Since the structure constants of the double graded commutator are also the structure constants of the symplectic 3-algebra, let us consider the following double graded commutator:

$$[\{Q_{\bar{i}\hat{i}}, Q_{\bar{j}\hat{j}}\}, Q_{\bar{k}k'}] = k\omega_{\hat{i}\hat{j}}(\delta_{\bar{j}\bar{k}} Q_{\bar{i}k'} - \delta_{\bar{i}\bar{k}} Q_{\bar{j}k'}). \tag{6.9}$$

It is not difficult to read off the structure constants of 3-algebra $f_{abc'd'}$:

$$f_{abc'd'} = f_{\bar{i}\hat{i},\bar{j}\hat{j},\bar{k}k',\bar{l}l'} = -k\omega_{\hat{i}\hat{j}}\omega_{k'l'}(\delta_{\bar{i}\bar{k}}\delta_{\bar{j}\bar{l}} - \delta_{\bar{i}\bar{l}}\delta_{\bar{j}\bar{k}}). \tag{6.10}$$

Note that $f_{abc'd'} = f_{\bar{i}\hat{i},\bar{j}\hat{j},\bar{k}k',\bar{l}l'}$ are not subjected to any linear constraint such as (4.12). To see this, let us consider the $Q_a Q_b Q_{c'}$ or $Q_{\bar{i}\hat{i}} Q_{\bar{j}\hat{j}} Q_{\bar{k}k'}$ Jacobi identity

$$[\{Q_{\bar{i}\hat{i}}, Q_{\bar{j}\hat{j}}\}, Q_{\bar{k}k'}] + [\{Q_{\bar{i}\hat{i}}, Q_{\bar{k}k'}\}, Q_{\bar{j}\hat{j}}] + [\{Q_{\bar{k}k'}, Q_{\bar{j}\hat{j}}\}, Q_{\bar{i}\hat{i}}] = 0, \tag{6.11}$$

which can be converted into

$$k[-\omega_{\hat{i}\hat{j}}\omega_{k'l'}(\delta_{\bar{i}\bar{k}}\delta_{\bar{j}\bar{l}} - \delta_{\bar{i}\bar{l}}\delta_{\bar{j}\bar{k}}) + \omega_{\hat{i}\hat{j}}\omega_{k'l'}(\delta_{\bar{i}\bar{k}}\delta_{\bar{j}\bar{l}} - \delta_{\bar{i}\bar{l}}\delta_{\bar{j}\bar{k}})]Q^{\bar{l}l'} = 0. \tag{6.12}$$

This equation is merely a statement that $f_{\bar{i}\hat{i},\bar{j}\hat{j},\bar{k}k',\bar{l}l'}$ are antisymmetric in $\bar{i}\bar{j}$ or in $\hat{i}\hat{j}$. Therefore the $Q_{\bar{i}\hat{i}} Q_{\bar{j}\hat{j}} Q_{\bar{k}k'}$ Jacobi identity does *not* impose a linear constraint on $f_{\bar{i}\hat{i},\bar{j}\hat{j},\bar{k}k',\bar{l}l'}$, since the two simple superalgebras share only *one* simple factor, as we claimed in the last section. Similarly, the $Q_{\bar{i}i'} Q_{\bar{j}j'} Q_{\bar{k}\hat{k}}$ Jacobi identity also does *not* impose a linear constraint on $f_{\bar{i}i',\bar{j}j',\bar{k}\hat{k},\bar{l}\hat{l}}$.

One can obtain $f_{abcd}$ by considering $[\{Q_{\bar{i}\hat{i}}, Q_{\bar{j}\hat{j}}\}, Q_{\bar{k}\hat{k}}]$. A short calculation gives

$$f_{abcd} = f_{\bar{i}\hat{i},\bar{j}\hat{j},\bar{k}\hat{k},\bar{l}\hat{l}} = -k[(\delta_{\bar{i}\bar{k}}\delta_{\bar{j}\bar{l}} - \delta_{\bar{i}\bar{l}}\delta_{\bar{j}\bar{k}})\omega_{\hat{i}\hat{j}}\omega_{\hat{k}\hat{l}} - \delta_{\bar{i}\bar{j}}\delta_{\bar{k}\bar{l}}(\omega_{\hat{i}\hat{k}}\omega_{\hat{j}\hat{l}} + \omega_{\hat{i}\hat{l}}\omega_{\hat{j}\hat{k}})]. \tag{6.13}$$

And $f_{a'b'c'd'}$ have a similar expression:

$$f_{a'b'c'd'} = f_{\bar{i}i',\bar{j}j',\bar{k}k',\bar{l}l'} = -k[(\delta_{\bar{i}\bar{k}}\delta_{\bar{j}\bar{l}} - \delta_{\bar{i}\bar{l}}\delta_{\bar{j}\bar{k}})\omega_{i'j'}\omega_{k'l'} - \delta_{\bar{i}\bar{j}}\delta_{\bar{k}\bar{l}}(\omega_{i'k'}\omega_{j'l'} + \omega_{i'l'}\omega_{j'k'})]. \tag{6.14}$$

Eqs. (6.10) $\sim$ (6.14) satisfy the symmetry conditions (4.10), the reality conditions (4.11) and the FIs (4.9). Eqs. (6.13) and (6.14) also satisfy the constraint equations (4.12). Substituting Eqs. (6.10) $\sim$ (6.14) into (4.37) and (4.38) gives the $\mathcal{N} = 4$ CSM theory with gauge group $Sp(2N_1) \times SO(N_2) \times Sp(2N_3)$.

Alternatively, one can read off $k_{uv}$ and $\tau^u_{ab}$ from (6.7) by comparing (6.7) with $[M^u, Q_a] = -\tau^u_{ab}\omega^{bc}Q_c$ and $\{Q_a, Q_b\} = \tau^u_{ab}k_{uv}M^v$. For instance,

$$\begin{aligned}(\tau_{\bar{m}\bar{n}})_{\bar{i}\hat{i},\bar{j}\hat{j}} &= \omega_{\hat{i}\hat{j}}(\delta_{\bar{m}\bar{i}}\delta_{\bar{n}\bar{j}} - \delta_{\bar{m}\bar{j}}\delta_{\bar{n}\bar{i}}), &(6.15)\\ k^{\bar{m}\bar{n},\bar{p}\bar{q}} &= \frac{k}{4}(\delta^{\bar{m}\bar{p}}\delta^{\bar{n}\bar{q}} - \delta^{\bar{m}\bar{q}}\delta^{\bar{n}\bar{p}}). &(6.16)\end{aligned}$$

Similarly, we have

$$(t_{\bar{p}\bar{q}})_{\bar{k}k',\bar{l}l'} = -\omega_{k'l'}(\delta_{\bar{p}\bar{k}}\delta_{\bar{q}\bar{l}} - \delta_{\bar{p}\bar{l}}\delta_{\bar{q}\bar{k}}). \tag{6.17}$$

Combining Eqs. (6.15) $\sim$ (6.17) gives (6.10):

$$f_{abc'd'} = k_{uv}\tau^u_{ab}t^v_{c'd'} = k^{\bar{m}\bar{n},\bar{p}\bar{q}}(\tau_{\bar{m}\bar{n}})_{\bar{i}\hat{i},\bar{j}\hat{j}}(t_{\bar{p}\bar{q}})_{\bar{k}k',\bar{l}l'} = f_{\bar{i}\hat{i},\bar{j}\hat{j},\bar{k}k',\bar{l}l'} = -k\omega_{\hat{i}\hat{j}}\omega_{k'l'}(\delta_{\bar{i}\bar{k}}\delta_{\bar{j}\bar{l}} - \delta_{\bar{i}\bar{l}}\delta_{\bar{j}\bar{k}}). \tag{6.18}$$

In this way, one also calculate $f_{abcd} = k_{uv}\tau^u_{ab}\tau^v_{cd}$ and $f_{a'b'c'd'} = k_{u'v'}\tau^{u'}_{a'b'}\tau^{v'}_{c'd'}$; they are the same as (6.13) and (6.14), respectively.

Analogously, we may set $N_1 = N_3$ but $N_2 \neq N_4$. In this case, the common simple factor is $\mathfrak{sp}(2N_1)$, and the gauge group is $SO(N_2) \times Sp(2N_1) \times SO(N_4)$.

However, if $N_1 = N_3$ and $N_2 = N_4$, then the two simple superalgebras are identical, or $\tau^v_{cd} = \tau^{v'}_{c'd'}$. As a result, the structure constants $f_{abc'd'} = f_{abcd}$, and $f_{abc'd'}$ also satisfy the constraint equation $f_{(abc')d'}$. In this case, the gauge group is $Sp(2N_1) \times SO(N_2)$, and the $\mathcal{N} = 4$ supersymmetry is enhanced to $\mathcal{N} = 5$. For details, see [35].

If $N_1 \neq N_3$ and $N_2 \neq N_4$, then $\{Q_a, Q_{c'}\} = 0$ and $f_{abc'd'} = 0$. Or equivalently, if the gauge group of the $\mathcal{N} = 4$ theory is $Sp(2N_1) \times SO(N_2) \times Sp(2N_3) \times SO(N_4)$, then the untwisted multiplets furnish a trivial representation of $Sp(2N_3) \times SO(N_4)$ and a fundamental representation of $Sp(2N_1) \times SO(N_2)$, while the twisted multiplets furnish a trivial representation of $Sp(2N_1) \times SO(N_2)$ and a fundamental representation of $Sp(2N_3) \times SO(N_4)$. In this case, the $\mathcal{N} = 4$ Lagrangian becomes two uncoupled GW Lagrangians (see section 6.2.3).

The theories of this section are first constructed in Ref. [34], using a different approach. We will rederive the general $\mathcal{N} = 4$ CSM theory in terms of ordinary Lie algebra in a forthcoming paper [45].

### 6.2.2 Examples of $\mathcal{N} = 4$ Quiver Gauge Theories

After working out the example in section 6.2.1, it is not difficult to find out the other gauge groups. We can consider the following pairs of super Lie algebras [35, 37]:

$$\begin{aligned}(G_1, G_2) \;=\; & (U(N_1|N_2), (U(N_2|N_3)), (OSp(N_1|2N_2), (OSp(N_1|2N_3)), \\ & (OSp(N_1|2N_2), (OSp(N_3|2N_2)), (OSp(N_1|2N_2), (OSp(2|2N_2)), \\ & (OSp(2|2N_1), (OSp(2|2N_1)). \qquad (6.19)\end{aligned}$$

For every pair, the even parts share at least one common simple factor, hence can be chosen as the Lie algebras of the gauge groups.

It is straightforward to generalize the construction of section 4 by decomposing the set of 3-algebra generators $T_I$ as *three* sets of generators, and decomposing one $\mathcal{N} = 5$ multiplet as three $\mathcal{N} = 4$ multiplets. Then the gauge group must be the even parts of $(G_1, G_2, G_3)$, where $G_i$ $(i = 1, 2, 3)$ is a super Lie algebra selected from the list (6.6). Here we assume that the even parts of $G_1$ and $G_2$ share at least one common simple factor, while the even parts of $G_2$ and $G_3$ share at least one common simple factor. For example, one can choose $(G_1, G_2, G_3)$ as $(OSp(N_1|2N_2), (OSp(N_1|2N_3), (OSp(N_4|2N_3))$. The resulting quiver diagram for gauge groups is

$$Sp(2N_2) - SO(N_1) - Sp(2N_3) - SO(N_4). \tag{6.20}$$

Or we can set $(G_1, G_2, G_3)$= $(U(N_1|N_2), (U(N_2|N_3)), (U(N_3|N_4)$, and the resulting quiver diagram for gauge groups is

$$U(N_1) - U(N_2) - U(N_3) - U(N_4). \tag{6.21}$$

In the general case, one can choose the even parts of $(G_1, \cdots, G_n)$, where $G_i$ $(i = 1, \cdots, n)$ is a super Lie algebra selected from the list (6.6); the even parts of $G_i$ and $G_{i+1}$ $(i = 1, \cdots, n-1)$ share one common simple factor [34, 37]. If the even parts of $G_1$ and $G_n$ (with $n$ an even number) also share one common simple factor, then the linear quiver becomes a closed loop. The linear quiver gauge theories described in this paragraph exhaust all known examples of $\mathcal{N} = 4$ superconformal CMS theories. As [37] pointed out, if one also takes account of the exceptional super Lie algebras, and the isomorphisms of the Lie algebras, there are additional possibilities. We will elaborate these ideas by constructing some $\mathcal{N} = 4$ theories with new gauge groups.

Let us first consider the exceptional Lie algebras. The even parts of the super groups $F(4)$, $G(3)$ and $D(2|1, \alpha)$ are $SO(7) \times SU(2)$ ($SO(7)$ is in the spinor representation), $G_2 \times SU(2)$ and $SO(4) \times Sp(2)$, respectively. We therefore may have

$$\begin{aligned}(G_1, G_2) \quad &= \quad (F(4), (SU(2|N_2)), (G(4), (SU(2|N_2)), (G(4), F(4)),\\ &\quad\quad (OSp(7|2N), F(4), (OSp(4|2N), D(2|1, \alpha)).\end{aligned} \tag{6.22}$$

Their even parts can be selected as the Lie algebras of the gauge groups.

It also is possible to construct some new $\mathcal{N} = 4$ CMS theories by using the four isomorphisms of the Lie algebras:

$$\mathfrak{so}(3) \cong \mathfrak{su}(2) \cong \mathfrak{sp}(2), \quad \mathfrak{so}(5) \cong \mathfrak{sp}(4), \quad \mathfrak{so}(6) \cong \mathfrak{su}(4). \tag{6.23}$$

The pairs of the super Lie algebras can be chosen as

$$\begin{aligned}(G_1, G_2) \quad = \quad & (OSp(3|2N_1), (OSp(N_2|2)), (OSp(3|N_1), (SU(2|N_2)), (OSp(3|2N_1), F_4),\\ & (OSp(3|2N_1), D(2|1, \alpha)), (OSp(3|2N_1), G_3), (OSp(N_1|2), (SU(2|N_2)),\\ & (OSp(N_1|2), F_4), (OSp(N_1|2), G_3), (G_3, D(2|1, \alpha)), (F_4, D(2|1, \alpha)),\\ & (OSp(5|2N_1), (OSp(N_2|4)), (OSp(6|N_1), (SU(4|N_2)),\end{aligned} \tag{6.24}$$

and their even parts can be selected as the Lie algebras of the gauge groups.

Clearly, one can use the constructions of the last two paragraphs in the general case $(G_1, \cdots, G_n)$, where we select $G_i$ $(i = 1, \cdots, n)$ from the list of the super Lie algebra (6.6). The even parts of $G_i$ and $G_{i+1}$ $(i = 1, \cdots, n-1)$ share one common simple factor; or one simple factor of the even part of $G_i$ is isomorphic to one simple factor of the even part of $G_{i+1}$.

Finally, one can obtain new gauge groups by noting that the even parts of $m$ $(m > 2)$ super Lie algebras can share one simple factor. One therefore can construct 'meshy' $\mathcal{N} = 4$ quiver gauge theories. For example, the even parts of the super Lie algebras $G_1 \sim G_4$ can share one common simple factor. For instance, if we set

$$(G_1, G_2, G_3, G_4) = (OSp(N|2N_1), OSp(N|2N_2), OSp(N|2N_3), OSp(N|2N_4)), \tag{6.25}$$

then the resulting quiver diagram for gauge groups is

$$\begin{array}{c} Sp(2N_1) \\ | \\ Sp(2N_2) - SO(N) - Sp(2N_4) \\ | \\ Sp(2N_3) \end{array} \tag{6.26}$$

The four multiplets are in the bifundamental representations of $Sp(2N_i) \times SO(N)$ $(i = 1, \cdots, 4)$, respectively. It can be seen that (6.26) is 'meshy', while (6.20) or (6.21) is 'linear'.

In summary, by using (6.19), (6.22) and (6.24), one can construct a general $\mathcal{N} = 4$ quiver gauge theory by requiring that the even parts of $a$ $(a \geq 2)$ *adjacent* super Lie algebras share one simple factor. (If two simple factors are isomorphic to each other, we also consider them as the same simple factor, even they may be in different representations.) The total number of the super Lie algebras is $n$ $(n \geq a)$, and the $n$ super Lie algebras are selected from (6.6). The Lie algebras of the gauge groups are just the even parts of the super Lie algebras, and the multiplets are in the bifundamental representations.

### 6.2.3 $\mathcal{N} = 4$ GW Theory in Terms of Lie Algebras

Here we consider the $\mathcal{N} = 4$ GW theory without the 'twisted' hyper multiplets, i.e., setting $\Phi^{a'}_{\dot{A}} = 0$. Then with the solution for structure constants of the 3-algebra given by

$$f_{abcd} = k_{mn}\tau^m_{ab}\tau^n_{cd}, \quad [\tau^m, \tau^n]_{ab} = C^{mn}{}_p\tau^p_{ab}, \tag{6.27}$$

which satisfy the FI as well as appropriate constraint and symmetry conditions, the gauge fields of the GW theory become

$$\tilde{A}_\mu{}^c{}_d \;=\; A_\mu^{ab} f_{ab}{}^c{}_d = A_\mu^{ab}\tau^m_{ab}k_{mn}\tau^{nc}{}_d \equiv A^m_\mu k_{mn}\tau^{nc}{}_d. \tag{6.28}$$

Following Ref. [33], we define the 'momentum map' and 'current ' operators as follows

$$\mu^m_{AB} \equiv \tau^m_{ab} Z^a_A Z^b_B, \quad j^m_{A\dot{B}} \equiv \tau^m_{ab} Z^a_A \psi^b_{\dot{B}}. \tag{6.29}$$

With Eqs (6.27) $\sim$ (6.29), Eqs. (4.40) and (4.41) become the Lagrangian and the supersymmetry law of the GW theory in Ref. [33], respectively:

$$\begin{aligned}\mathcal{L} \;=\;& \frac{1}{2}\epsilon^{\mu\nu\lambda}(k_{mn}A^m_\mu\partial_\nu A^n_\lambda + \frac{1}{3}\tilde{C}_{mnp}A^m_\mu A^n_\nu A^p_\lambda) + \frac{1}{2}(-D_\mu\bar{Z}^A_a D^\mu Z^a_A + i\bar{\psi}^{\dot{A}}_a\gamma^\mu D_\mu\psi^a_{\dot{A}})\\ &-\frac{i}{2}k_{mn}j^m_{A\dot{B}}j^{nA\dot{B}} - \frac{1}{24}\tilde{C}_{mnp}\mu^{mA}{}_B\mu^{nB}{}_C\mu^{pC}{}_A,\end{aligned} \tag{6.30}$$

with $\tilde{C}_{mnp} = k_{mr}k_{ns}C^{rs}{}_p$ and

$$\begin{aligned}&\delta Z^a_A = i\epsilon_A{}^{\dot{A}}\psi^a_{\dot{A}},\\ &\delta\psi^a_{\dot{A}} = -\gamma^\mu D_\mu Z^a_B \epsilon^{\dagger}_{\dot{A}}{}^B - \frac{1}{3}k_{mn}\tau^{ma}{}_b Z^b_B\mu^{nB}{}_C\epsilon^{\dagger}_{\dot{A}}{}^C,\\ &\delta A^m_\mu = i\epsilon^{A\dot{B}}\gamma_\mu j^m_{A\dot{B}}.\end{aligned} \tag{6.31}$$

Since we derived the GW theory by decomposing the $\mathcal{N} = 5$ theory and setting the twisted mulitplets to zero, so the classical superalgebras, which are used to realize the 3-algebra, must be the same as those used in the $\mathcal{N} = 5$ case, i.e.,

$$U(M|N), \quad OSp(M|2N), \quad OSp(2|2N), \quad F(4), \quad G(3), \quad D(2|1;\alpha). \tag{6.32}$$

Indeed, they are of the same form as that of the superalgebra (5.2). Therefore their bosonic parts can be selected to be the Lie algebras of the gauge groups of the GW theory [33, 34, 37]; and the corresponding representations are determined by the fermionic generators.

# CHAPTER 7

# $\mathcal{N} = 6, 8$ CSM THEORIES AND 3-ALGEBRAS

In Ref. [35], the $\mathcal{N} = 6$ theories are derived from the $\mathcal{N} = 5$ theories by enhancing the R-symmetry from $Sp(4)$ to $SU(4)$. In this section we will implement the same idea in the context of 3-algebras [40]. We will call the symplectic 3-algebras presented in Ref. [40] and in [38], respectively, to construct the $\mathcal{N} = 5$, $\mathcal{N} = 6$ theories as the "$\mathcal{N} = 5$, $\mathcal{N} = 6$ three-algebra", respectively. We will see that the symplectic 3-algebra provides a framework unifying the $\mathcal{N} = 4, 5, 6, 8$ CSM theories.

## 7.1 General $\mathcal{N}$=6 CSM Theories

The enhancement of R-symmetry from $Sp(4)$ to $SU(4)$ in Ref. [35] is based on the following observation: The reality condition (3.2) implies that the complex conjugates of the matter fields can be obtained by similarity transformations, i.e.,

$$\bar{Z}_I^A = \omega^{AB}\omega_{IJ}Z_B^J, \quad \psi_I^A = \omega^{AB}\omega_{IJ}\psi_B^J. \tag{7.1}$$

Therefore the matter fields actually furnish a pseudo-real presentation of the gauge group. If we decompose this pseudo-real representation into a complex representation and its conjugate representation, then the $Sp(4)$ R-symmetry will be enhanced to $SU(4)$, and the global $\mathcal{N} = 5$ SUSY will get enhanced to $\mathcal{N} = 6$.

In this section, we will show that this enhancement can be implemented exclusively in the framework of symplectic 3-algebra, which thus provides a unified framework for both $\mathcal{N} = 5$ and $\mathcal{N} = 6$ theories. Since in our approach the ordinary Lie algebra of the gauge groups is generated by the FI and the 3-brackets, the challenge we face is to derive the $\mathcal{N} = 6$ three-algebra from the 3-algebra proposed in Ref. [40].

Following Ref. [35], we first decompose an $\mathcal{N} = 5$ scalar field as a direct sum of an $\mathcal{N} = 6$ scalar field and its complex conjugate (See Eq. (7.6)):

$$Z_A^I = Z_A^{a\alpha} = \bar{Z}_A^a\chi_{1\alpha} + \omega_{AB}Z_a^B\chi_{2\alpha} = \bar{Z}_A^a\delta_{1\alpha} + \omega_{AB}Z_a^B\delta_{2\alpha}, \tag{7.2}$$

where the right hand side of the arrow contains $\mathcal{N}=6$ fields. Here the index $I$ runs from 1 to $2L$, while the index $a$ runs from 1 to $L$. And $\chi_{1\alpha}$ and $\chi_{2\alpha}$ are "spin up" and "spin down" spinor, respectively, i.e., [1]

$$\chi_{1\alpha}=\begin{pmatrix}1\\0\end{pmatrix}\ ,\qquad \chi_{2\alpha}=\begin{pmatrix}0\\1\end{pmatrix}. \tag{7.3}$$

To make the $\mathcal{N}=5$ SUSY transformation law (3.45) consistent with that of $\mathcal{N}=6$ (see below the first two equations of Eq. (7.22)), we have to decompose the $\mathcal{N}=5$ fermion fields as follows:

$$\psi^I_A=\psi^{a\alpha}_A=\omega_{AB}\psi^{Ba}\delta_{1\alpha}-\psi_{Aa}\delta_{2\alpha}, \tag{7.4}$$

where the right hand side contains $\mathcal{N}=6$ fermion fields. We further decompose the antisymmetric tensor $\omega_{IJ}$ and its inverse as

$$\begin{aligned}\omega_{IJ}=\omega_{a\alpha,b\beta}&=\delta_a{}^b\delta_{1\alpha}\delta_{2\beta}-\delta^a{}_b\delta_{2\alpha}\delta_{1\beta},\\ \omega^{IJ}=\omega^{a\alpha,b\beta}&=\delta_a{}^b\delta_{2\alpha}\delta_{1\beta}-\delta^a{}_b\delta_{1\alpha}\delta_{2\beta}\ .\end{aligned} \tag{7.5}$$

Then the reality condition (7.1) reads

$$Z^{*A}_a=\bar{Z}^a_A,\qquad \psi^{*Aa}=\psi_{Aa}, \tag{7.6}$$

in agreement with those for $\mathcal{N}=6$ theories. This justifies the above decomposition (7.5) of the antisymmetric tensor of the $\mathcal{N}=5$ three-algebra to derive the $\mathcal{N}=6$ three-algebra.

To be compatible with the decomposition of scalar and fermion fields, one has to decompose the gauge fields as

$$\tilde{A}_\mu{}^I{}_J=\tilde{A}_\mu{}^{a\alpha}{}_{b\beta}=\tilde{A}_\mu{}^a{}_b\delta_{1\alpha}\delta_{1\beta}-\tilde{A}_\mu{}^b{}_a\delta_{2\alpha}\delta_{2\beta}, \tag{7.7}$$

where the right hand side is a direct sum of an $\mathcal{N}=6$ gauge field and its complex conjugate. Since our gauge fields $\tilde{A}_\mu{}^K{}_L$ are defined in terms of the structure constants of a 3-algebra, i.e.,

$$\tilde{A}_\mu{}^K{}_L=A^{IJ}_\mu f_{IJ}{}^K{}_L, \tag{7.8}$$

we have to decompose its structure constants properly to result in the desired decomposition Eq. (7.7). We find that Eq. (7.7) indeed follows from the decomposition of the structure constants given by

$$\begin{aligned}f_{IJKL}=f_{a\alpha,b\beta,c\gamma,d\delta}\ &=\ f^{ac}{}_{db}\delta_{2\alpha}\delta_{1\beta}\delta_{2\gamma}\delta_{1\delta}+f^{ad}{}_{cb}\delta_{2\alpha}\delta_{1\beta}\delta_{1\gamma}\delta_{2\delta}\\ &\quad+f^{bc}{}_{da}\delta_{1\alpha}\delta_{2\beta}\delta_{2\gamma}\delta_{1\delta}+f^{bd}{}_{ca}\delta_{1\alpha}\delta_{2\beta}\delta_{1\gamma}\delta_{2\delta},\end{aligned} \tag{7.9}$$

[1] Here the index $\alpha$ is *not* an index of a spacetime spinor. We hope this will not cause any confusion.

combined with the decomposition of $A_\mu^{IJ}$ given by

$$A_\mu^{IJ} = A_\mu^{a\alpha,b\beta} = -\frac{1}{2}(A_\mu{}^a{}_b\delta_{1\alpha}\delta_{2\beta} + A_\mu{}^b{}_a\delta_{2\alpha}\delta_{1\beta}). \tag{7.10}$$

With these decompositions, the $\mathcal{N} = 6$ gauge fields become: (see the right side of Eq. (7.7))

$$\tilde{A}_\mu{}^c{}_d = A_\mu{}^b{}_a f^{ca}{}_{bd}. \tag{7.11}$$

Later we will identify the above $f^{ca}{}_{bd}$ in the right side of Eq. (7.9) as the structure constants of the $\mathcal{N} = 6$ three-algebra. With Eq. (7.5) and (7.9), the reality condition of the structure constants (2.15) reduces to

$$f^{*ab}{}_{cd} = f^{cd}{}_{ab}, \tag{7.12}$$

as desired for the $\mathcal{N} = 6$ three-algebra [38, 39].

Eq. (7.5) motivates us to decompose the generators of the 3-algebra as follows:

$$\begin{aligned} T_I = T_{a\alpha} &= \omega_{a\alpha,b\beta}T^{b\beta} \\ &= T^{a2}\delta_{1\alpha} - T^{a1}\delta_{2\alpha}. \end{aligned} \tag{7.13}$$

Since we decompose a matter field as a direct sum of a $\mathcal{N} = 6$ matter field and its *complex conjugate*, it is necessary to decompose a generator of the 3-algebra as a direct sum of a generator of a 3-algebra and its *complex conjugate*. This can be accomplished by setting

$$t^a = T^{a1}, \quad \bar{t}_a \equiv t^{*a} = T^{a2}, \tag{7.14}$$

where $t^a$ is a generator of the 3-algebra, and $\bar{t}_a$ its complex conjugate.

The hermitian bilinear form of two $\mathcal{N} = 5$ fields will be (for instance):

$$\begin{aligned} Z^{*I}_{1A}Z^I_{2A} &= \omega_{IJ}\omega^{AB}Z^J_{1B}Z^I_{2A} \\ &= \omega_{a\alpha,b\beta}\omega^{AB}Z^{b\beta}_{1B}Z^{a\alpha}_{2A} \\ &= \bar{Z}^a_{2A}Z^A_{1a} + \bar{Z}^a_{1A}Z^A_{2a}. \end{aligned} \tag{7.15}$$

Namely, it becomes a sum of the hermitian bilinear form of two $\mathcal{N} = 6$ fields and its complex conjugate. Generally speaking, the hermitian bilinear form of two arbitrary $\mathcal{N} = 6$ three-algebra valued fields will become

$$h(X, Y) = X^*_aY_a \equiv \bar{X}^aY_a. \tag{7.16}$$

The reality condition (7.12) and Eq. (7.9) imply that the $\mathcal{N}=5$ three-bracket (2.1) can be decomposed as a direct sum of $\mathcal{N}=6$ brackets and their complex conjugates as follows:

$$\begin{aligned}
[T_I, T_J; T_K] &= [T_{a\alpha}, T_{b\beta}; T_{c\gamma}] \\
&= [t^a, t^c; \bar{t}_b]\delta_{2\alpha}\delta_{1\beta}\delta_{2\gamma} + [t^a, t^c; \bar{t}_b]^*\delta_{1\alpha}\delta_{2\beta}\delta_{1\gamma} \\
&\quad +[t^b, t^c; \bar{t}_a]\delta_{1\alpha}\delta_{2\beta}\delta_{2\gamma} + [t^b, t^c; \bar{t}_a]^*\delta_{2\alpha}\delta_{1\beta}\delta_{1\gamma}.
\end{aligned} \tag{7.17}$$

Here the 3-brackets

$$[t^a, t^c; \bar{t}_b] = f^{ac}{}_{bd} t^d. \tag{7.18}$$

are those for the $\mathcal{N}=6$ three-algebra. Such 3-brackets were first proposed by Bagger and Lambert [38] for a $\mathcal{N}=6$ CSM theory. An unusual feature of the 3-brackets is that it involves complex conjugate for the third generator. Our above decomposition from the $\mathcal{N}=5$ three-algebra reveals clearly the origin of the need for complex conjugation of the third generator.

Later we will see that the structure constants defined in Eq. (7.18) are indeed *anti-symmetric* in the first two indices. (See Eq. (7.20).) With Eq. (7.17), the fundamental identity (2.6) reduces to

$$f^{fc}{}_{dg} f^{ag}{}_{eb} - f^{af}{}_{gb} f^{gc}{}_{de} + f^{cf}{}_{eg} f^{ag}{}_{db} - f^{ac}{}_{gb} f^{gf}{}_{ed} = 0, \tag{7.19}$$

as desired. Also the constraint condition (2.17) on the structure constants and the symmetry properties (2.18) of the structure constants reduce to

$$f^{ab}{}_{cd} = -f^{ba}{}_{cd} = f^{ba}{}_{dc}. \tag{7.20}$$

One easily recognizes that eqs. (7.16), (7.18), (7.19), (7.12), and (7.20) are those defining the $\mathcal{N}=6$ three-algebra used in Ref. [38]. (The relation between the $\mathcal{N}=6$ three-algebra and super Lie algebra was discussed in Ref. [46].)

Substituting Eq. (7.2), (7.4), (7.9), and (7.10) into the $\mathcal{N}=5$ Lagrangian (3.38) and the SUSY transformation law (3.45), and using the $Sp(4)$ identity (A.34) and (A.35), we reproduce the $\mathcal{N}=6$ Lagrangian

$$
\begin{aligned}
\mathcal{L} \;=\; & -D_\mu \bar{Z}^a_A D^\mu Z^A_a - i\bar{\psi}^{Aa}\gamma^\mu D_\mu \psi_{Aa} \\
& -if^{ab}{}_{cd}\bar{\psi}^{Ad}\psi_{Aa}Z^B_b\bar{Z}^c_B + 2if^{ab}{}_{cd}\bar{\psi}^{Ad}\psi_{Ba}Z^B_b\bar{Z}^c_A \\
& -\frac{i}{2}\varepsilon_{ABCD}f^{ab}{}_{cd}\bar{\psi}^{Ac}\psi^{Bd}Z^C_a Z^D_b - \frac{i}{2}\varepsilon^{ABCD}f^{cd}{}_{ab}\bar{\psi}_{Ac}\psi_{Bd}\bar{Z}^a_C\bar{Z}^b_D \\
& +\frac{1}{2}\varepsilon^{\mu\nu\lambda}(f^{ab}{}_{cd}A_\mu{}^c{}_b\partial_\nu A_\lambda{}^d{}_a + \frac{2}{3}f^{ac}{}_{dg}f^{ge}{}_{fb}A_\mu{}^b{}_a A_\nu{}^d{}_c A_\lambda{}^f{}_e) \\
& -\frac{2}{3}(f^{ab}{}_{cd}f^{ed}{}_{fg} - \frac{1}{2}f^{eb}{}_{cd}f^{ad}{}_{fg})\bar{Z}^c_A Z^A_e \bar{Z}^f_B Z^B_a \bar{Z}^g_D Z^D_b,
\end{aligned} \tag{7.21}
$$

and the $\mathcal{N}=6$ SUSY transformation law reads

$$
\begin{aligned}
\delta Z^A_d &= -i\bar{\epsilon}^{AB}\psi_{Bd} \\
\delta \bar{Z}^d_A &= -i\bar{\epsilon}_{AB}\psi^{Bd} \\
\delta\psi_{Bd} &= \gamma^\mu D_\mu Z^A_d \epsilon_{AB} + f^{ab}{}_{cd}Z^C_a Z^A_b \bar{Z}^c_C \epsilon_{AB} + f^{ab}{}_{cd}Z^C_a Z^D_b \bar{Z}^c_B \epsilon_{CD} \\
\delta\psi^{Bd} &= \gamma^\mu D_\mu \bar{Z}^d_A \epsilon^{AB} + f^{cd}{}_{ab}\bar{Z}^a_C \bar{Z}^b_A Z^C_c \epsilon^{AB} + f^{cd}{}_{ab}\bar{Z}^a_C \bar{Z}^b_D Z^B_c \epsilon^{CD} \\
\delta\tilde{A}_\mu{}^c{}_d &= -i\bar{\epsilon}_{AB}\gamma_\mu Z^A_a \psi^{Bb} f^{ca}{}_{bd} + i\bar{\epsilon}^{AB}\gamma_\mu \bar{Z}^a_A \psi_{Bb} f^{cb}{}_{ad}.
\end{aligned} \tag{7.22}
$$

Here the SUSY transformation parameters $\epsilon_{AB}$ satisfy

$$
\epsilon_{AB} = -\epsilon_{BA} \tag{7.23}
$$

$$
\epsilon^*_{AB} = \epsilon^{AB} = \frac{1}{2}\varepsilon^{ABCD}\epsilon_{CD} \tag{7.24}
$$

Now the parameters $\epsilon_{AB}$ transform as the **6** of $SU(4)$. It is in this sense that the global $\mathcal{N}=5$ SUSY gets enhanced to $\mathcal{N}=6$. The Lagrangian (7.21) and the transformation law (7.22) are the same as the ones obtained in the 3-algebra approach for $\mathcal{N}=6$ theories in Ref. [38].

The $\mathcal{N}=6$ superconformal CSM theories in three dimensions can be classified by super Lie algebras [33, 35, 49] or by using group theory [48]. Two primary types are allowed: with gauge group $U(M)\times U(N)$ and $Sp(2N)\times U(1)$, respectively. In section 7.4, we will drive these two theories by specifying the structure constants of the $\mathcal{N}=6$ three-algebra.

## 7.2 $\mathcal{N}$=8 CSM Theory

If the inner product (7.16) becomes the standard inner product in the Euclidian space

$$
h(X,Y) = X_a Y_a \quad \text{or} \quad h(t^a, t^b) = \delta^{ab}, \tag{7.25}
$$

then there is no difference between a lower index $a$ and an upper index $a$, i.e., $\bar{t}_a = \bar{t}^a$. As a result, the 3-bracket (7.18) becomes

$$[t^a, t^c, \bar{t}^b] = f^{acb}{}_d t^d. \tag{7.26}$$

If the first 3 indices of $f^{acb}{}_d$ are antisymmetric, then Eq. (7.26) becomes the famous Nambu bracket. And the reality condition (7.12) becomes

$$f^{ab}{}_{cd} = f^{cd}{}_{ab}. \tag{7.27}$$

The symmetry properties of the structure constants (7.20) imply that

$$f^{abcd} \equiv \delta^{de} f^{abc}{}_e \tag{7.28}$$

are *totally antisymmetric*. Now the FI (7.19) can be converted into

$$f^{afe}{}_g f^{cdg}{}_b - f^{cda}{}_g f^{gfe}{}_b - f^{cdf}{}_g f^{age}{}_b - f^{cde}{}_g f^{afg}{}_b = 0. \tag{7.29}$$

The 3-algebra defined by Eq. (7.26) $\sim$ (7.29) is nothing but the Nambu 3-algebra. Substituting Eq. (7.28) into (7.21) and (7.22) gives the BLG theory, since the $\mathcal{N} = 6$ supersymmetry is promoted to $\mathcal{N} = 8$ if the structure constants are totally antisymmetric [47].

To demonstrate that the Nambu 3-algebra does generate an $SO(4)$ gauge group, we choose the following 4 $\sigma$-matrices (the first three are Pauli matrices)

$$\sigma^a = (\sigma^1, \sigma^2, \sigma^3, i\mathbb{I}) \tag{7.30}$$

to realize the generators of the Nambu 3-algebra [47], i.e.,

$$t^a \doteq \sigma^a \quad \text{and} \quad \bar{t}^a \doteq \sigma^{a\dagger}, \tag{7.31}$$

where $\sigma^{a\dagger}$ is the hermitian conjugate of $\sigma^a$. It is well known that one can establish a connection between the $SU(2) \times SU(2)$ and $SO(4)$ group by (7.30). These $\sigma$-matrices satisfy the Clifford algebra:

$$\sigma^a \sigma^{b\dagger} + \sigma^b \sigma^{a\dagger} = 2\delta^{ab} \quad \text{and} \quad \sigma^{a\dagger}\sigma^b + \sigma^{b\dagger}\sigma^a = 2\delta^{ab}. \tag{7.32}$$

The inner product (7.25) is defined as:

$$h(\sigma^a, \sigma^b) = \frac{1}{2}\mathrm{Tr}(\sigma^{a\dagger}\sigma^b) = \delta^{ab}, \tag{7.33}$$

where we have normalized the trace by a factor $\frac{1}{2}$. We specify the 3-bracket as [38]:

$$
\begin{aligned}
[\sigma^a, \sigma^b; \sigma^{c\dagger}] &= k(\sigma^a \sigma^{c\dagger} \sigma^b - \sigma^b \sigma^{c\dagger} \sigma^a) \\
&= -2k\varepsilon^{abcd}\sigma_d,
\end{aligned}
$$

where $\varepsilon^{abcd}$ is the familiar Levi-Civita tensor. So in this realization, the structure constants $f^{abcd}$ are nothing but $\varepsilon^{abcd}$ (up to an unimportant constant). And from the point of view of ordinary Lie algebra, a field valued in the Nambu 3-algebra

$$
Z^A{}_\alpha{}^{\dot\alpha} = Z^A_a \sigma^a{}_\alpha{}^{\dot\alpha} \quad (a = 1, \cdots, 4; \quad \alpha, \dot\alpha = 1, 2.)
$$

is indeed in the bifundamental representation of $SU(2) \times SU(2)$. The $\mathcal{N} = 8$ BLG theory is essentially unique, since the Nambu 3-algebra with a symmetric and positive define metric can only generate an $SO(4)$ gauge symmetry [20, 21]. In section (8.2), we will demonstrate that the Nambu 3-algebra can be realized in terms of a super Lie algebra $PSU(2|2)$.

## 7.3 Closure of the $\mathcal{N} = 6$ Algebra

We require the on-shell closure of the supersymmetry algebra. Namely, after imposing equations of motion, the commutator of two supersymmetry transformations must be equal to a translation plus a gauge term.

The commutator of two supersymmetry transformations acting on the scalar fields reads [38, 39]

$$
[\delta_1, \delta_2] Z^A_d = v^\mu \partial_\mu Z^A_d + (\tilde\Lambda^a{}_d - v^\mu \tilde A_\mu{}^a{}_d) Z^A_a, \tag{7.34}
$$

where

$$
\begin{aligned}
v^\mu &= \frac{i}{2} \bar\epsilon_2^{CD} \gamma^\mu \epsilon_{1CD}, && (7.35)\\
\tilde\Lambda^a{}_d &= \Lambda^c{}_b f^{ab}{}_{cd}, && (7.36)\\
\Lambda^c{}_b &= i(\bar\epsilon_2^{DE} \epsilon_{1CE} - \bar\epsilon_1^{DE} \epsilon_{2CE}) \bar Z^c_D Z^C_b. && (7.37)
\end{aligned}
$$

The first term of Eq. (7.34) is a translation, and the second represents a gauge transformation, as expected. In deriving (7.34), we have used Eq. (7.20): $f^{ab}{}_{cd} = -f^{ba}{}_{cd}$.

For the gauge field, using the FI (7.19) and some identities in Appendix A.5, we obtain

$$\begin{aligned}[][\delta_1, \delta_2]\tilde{A}_\mu{}^c{}_d &= v^\nu \partial_\nu \tilde{A}_\mu{}^c{}_d + D_\mu(\tilde{\Lambda}^c{}_d - v^\nu \tilde{A}_\nu{}^c{}_d) \\ &+ v^\nu \left[ \tilde{F}_{\mu\nu}{}^c{}_d + \varepsilon_{\mu\nu\lambda} \left( D^\lambda Z^A_a \bar{Z}^b_A - Z^A_a D^\lambda \bar{Z}^b_A - i\bar{\psi}^{Ab}\gamma^\lambda \psi_{Aa} \right) f^{ac}{}_{bd} \right]. \end{aligned} \tag{7.38}$$

where $\tilde{F}_{\mu\nu}{}^c{}_d = \partial_\mu \tilde{A}_\nu{}^c{}_d - \partial_\nu \tilde{A}_\mu{}^c{}_d + [\tilde{A}_\mu, \tilde{A}_\nu]^c{}_d$ is the field strength. We recognize the first term as a translation, and the second a gauge transformation. To achieve the closure, we need to impose the following equation of motion for the gauge field:

$$\tilde{F}_{\mu\nu}{}^c{}_d = -\varepsilon_{\mu\nu\lambda} \left( D^\lambda Z^A_a \bar{Z}^b_A - Z^A_a D^\lambda \bar{Z}^b_A - i\bar{\psi}^{Ab}\gamma^\lambda \psi_{Aa} \right) f^{ac}{}_{bd}. \tag{7.39}$$

As BL discovered [38], the FI implies $D_\mu f^{ca}{}_{bd} = 0$, if one writes $\tilde{A}_\mu{}^c{}_d = A_\mu{}^b{}_a f^{ca}{}_{bd}$ in the expression of the covariant derivative. We have used this important equation to derive the second term in Eq. (7.38): $f^{ca}{}_{bd} D_\mu \Lambda^b{}_a = D_\mu \tilde{\Lambda}^c{}_d$.

The commutator of two supersymmetry transformations acting on the fermionic fields reads

$$\begin{aligned}[][\delta_1, \delta_2]\psi_{Dd} &= v^\mu \partial_\mu \psi_{Dd} + (\tilde{\Lambda}^a{}_d - v^\mu \tilde{A}_\mu{}^a{}_d)\psi_{Da} \\ &- \frac{i}{2}(\bar{\epsilon}_1^{AC}\epsilon_{2AD} - \bar{\epsilon}_2^{AC}\epsilon_{1AD})E_{Cd} \\ &+ \frac{i}{4}(\bar{\epsilon}_1^{AB}\gamma_\nu \epsilon_{2AB})\gamma^\nu E_{Dd}, \end{aligned} \tag{7.40}$$

where

$$E_{Cd} = \gamma^\mu D_\mu \psi_{Cd} + f^{ab}{}_{cd}\left( \psi_{Ca} Z^D_b \bar{Z}^c_D - 2\psi_{Da} Z^D_b \bar{Z}^c_C - \varepsilon_{CDEF}\psi^{Dc} Z^E_a Z^F_b \right). \tag{7.41}$$

Again, the first two term are a translation and a gauge transformation, respectively. To achieve the closure of the supersymmetry algebra, we have to impose the following equations of motion for the fermionic fields:

$$0 = E_{Cd} = \gamma^\mu D_\mu \psi_{Cd} + f^{ab}{}_{cd}\left( \psi_{Ca} Z^D_b \bar{Z}^c_D - 2\psi_{Da} Z^D_b \bar{Z}^c_C - \varepsilon_{CDEF}\psi^{Dc} Z^E_a Z^F_b \right). \tag{7.42}$$

To derive the equations of motion of the scalar fields, we take the super-variation of the equations of motion of the fermionic fields: $\delta E_{Cd} = 0$. Two equations are obtained: One is

$$\begin{aligned} 0 &= D_\mu D^\mu Z^B_c - i f^{ab}{}_{cd}(\bar{\psi}^{Ad}\psi_{Aa} Z^B_b - 2\bar{\psi}^{Bd}\psi_{Aa} Z^A_b - \varepsilon^{ABCD}\bar{\psi}_{Aa}\psi_{Cb}\bar{Z}^d_D) \\ &+ \frac{1}{3}(f^{ae}{}_{fd} f^{bd}{}_{cg} - 2f^{ab}{}_{cd} f^{ed}{}_{fg} - 2f^{db}{}_{gc} f^{ae}{}_{fd} + 2f^{ab}{}_{fd} f^{ed}{}_{cg} - 4f^{eb}{}_{fd} f^{ad}{}_{cg}) \\ &\times Z^B_e \bar{Z}^f_A Z^A_a \bar{Z}^g_D Z^D_b. \end{aligned} \tag{7.43}$$

The other equation is equivalent to the equation of motion of the gauge field (7.39).

The equations of motion of the gauge, fermion and scalar fields, Eqs. (7.39), (7.42) and (7.43), respectively, can be derived from the Lagrangian (7.21).

## 7.4 Examples of the $\mathcal{N}=6$ Theories

### 7.4.1 $\mathcal{N}=6$, $Sp(2N)\times U(1)$

We first specify the structure constants as [2] [39]

$$f_{a-,b-,c+,d+} = -k[(\omega_{ab}\omega_{cd}+\omega_{ac}\omega_{bd})h_{-+}h_{-+}+(\omega_{ad}\epsilon_{-+})(\omega_{bc}\epsilon_{-+})], \tag{7.44}$$

where $k$ is a real constant, $\omega^{ab}$ an antisymmetric bilinear form $(a,b=1,2,\cdots,2N)$, $h_{+-}=h_{-+}=1$ and $\epsilon_{+-}=-\epsilon_{-+}=ih_{+-}$. Here $a,b$ are the $Sp(2N)$ indices while $+,-$ the $SO(2)$ indices. We use the gauge invariant antisymmetric tensor $\omega^{a+,b-}\equiv\omega^{ab}h^{+-}$ to raise the first two pairs of indices of the structure constants (7.44):

$$f^{a+b+}{}_{c+d+} = k[(\omega^{ab}\omega_{cd}-\delta^a{}_c\delta^b{}_d)\delta^+{}_+\delta^+{}_+-(\delta^a{}_d)(-i\delta^+{}_+)(\delta^b{}_c)(-i\delta^+{}_+)]. \tag{7.45}$$

Suppressing the $SO(2)$ indices gives

$$f^{ab}{}_{cd} = k(\omega^{ab}\omega_{cd}+\delta^a{}_d\delta^b{}_c-\delta^a{}_c\delta^b{}_d). \tag{7.46}$$

It is not too difficult to check that the structure constants satisfy the FI (7.19) and the reality condition (7.12), and also have the desired symmetry properties (7.20).

In fact, in accordance with Eq. (7.11) and (7.46), the gauge fields can be decomposed into two parts:

$$\begin{aligned}\tilde{A}_\mu{}^c{}_d &= A_\mu{}^b{}_a f^{ca}{}_{bd} \\ &= -(A_{\mu d}{}^c + A_\mu{}^c{}_d)+(A_\mu{}^a{}_a)\delta^c{}_d \\ &\equiv B_\mu{}^c{}_d + A_\mu\delta^c{}_d.\end{aligned} \tag{7.47}$$

It is natural to identify the trace part $A_\mu\equiv A_\mu{}^a{}_a$ as the $U(1)$ part of the gauge potential, and $B_\mu{}^c{}_d$ as the $Sp(2N)$ part. The reason is that we can recast $B_\mu{}^c{}_d$ as $A_\mu^{ab}(t_{ab})^c{}_d$, where $(t_{ab})^c{}_d$ is in the fundamental representation of the Lie algebra of $Sp(2N)$. Therefore the gauge group is nothing but $Sp(2N)\times U(1)$, whose Lie algebra is the bosonic part of the super Lie algebra $OSp(2|2N)$. In section 8.1, we will use $OSp(2|2N)$ to realize the 3-algebra used in this subsection.

---

[2] In the Lagrangian (7.21) of section 7.1, the index $a$ runs from 1 to $L$. In this subsection, we split it into two indices: $a\to a\pm$, and set $L=4N$. We hope this will not cause any confusion.

We substitute the structure constants (7.46) into (7.22). We then obtain the $\mathcal{N}=6$ (on-shell) SUSY transformation law in the theory (see Appendix C.2.1). The equations of motion can be derived from the Lagrangian obtained by substituting Eq. (7.46) into the Lagrangian (7.21) and replacing $A_\mu{}^b{}_a$ by $\frac{1}{k}A_\mu{}^b{}_a$ (see Appendix C.2.1). The SUSY transformation law (C.23) and the Lagrangian (C.22) are indeed in agreement with the $\mathcal{N}=6, Sp(2M)\times U(1)$ superconformal CSM theory derived from the ordinary Lie algebra in Ref. [35].

### 7.4.2 $\mathcal{N}=6$, $U(M)\times U(N)$

The Lagrangian of this theory has been constructed in Ref. [38]. For this thesis to be self-contained, it is worth presenting the Lagrangian and SUSY transformation law of $D=3, \mathcal{N}=6$, $U(M)\times U(N)$ theory in this subsection.

To generate a direct gauge group such as $U(M)\times U(N)$, we split up a *lower* 3-algebra index $a$ into two indices: $a\to n\hat{n}$, where $n=1,...,M$ is a fundamental index of $U(M)$, $\hat{n}=1,...,N$ an antifundamental index of $U(N)$. With this decomposition, the hermitian inner product (7.16) can be written as a trace:

$$X^*_aY_a \to X^*_{n\hat{n}}Y_{n\hat{n}} = X^{*\mathrm{t}}_{\hat{n}n}Y_{n\hat{n}} \equiv \mathrm{Tr}(X^\dagger Y), \tag{7.48}$$

where the superscript "t" stands for the usual transpose. On the other hand, according to the definition (7.16), the hermitian inner product can be also written as: $X^*_aY_a \equiv \bar{X}^aY_a$, which leads us to decompose an *upper* index $a$ as $a\to \hat{n}n$. Thus the hermitian inner product can be written as

$$X^*_aY_a \equiv \bar{X}^aY_a \to \bar{X}^{\hat{n}n}Y_{n\hat{n}} \equiv \mathrm{Tr}(\bar{X}Y) = \mathrm{Tr}(X^\dagger Y). \tag{7.49}$$

We then specify the 3-bracket (7.18) to be

$$[t^{\hat{k}k}, t^{\hat{l}l}; \bar{t}_{m\hat{m}}] = k(\delta^{\hat{k}}{}_{\hat{m}}\delta^l{}_m t^{\hat{l}k} - \delta^{\hat{l}}{}_{\hat{m}}\delta^k{}_m t^{\hat{k}l}). \tag{7.50}$$

The structure constants can be easily read off as

$$f^{\hat{k}k,\hat{l}l}{}_{m\hat{m},n\hat{n}} = k(\delta^{\hat{k}}{}_{\hat{m}}\delta^{\hat{l}}{}_{\hat{n}}\delta^k{}_n\delta^l{}_m - \delta^{\hat{k}}{}_{\hat{n}}\delta^{\hat{l}}{}_{\hat{m}}\delta^k{}_m\delta^l{}_n). \tag{7.51}$$

It is straightforward to check that the structure constants $f^{\hat{k}k,\hat{l}l}{}_{m\hat{m},n\hat{n}}$ satisfy the FI (7.19) and the reality conditions (7.12), and has the symmetry properties (7.20). The structure constants are first discovered by BL [38] (though they did not write down Eq. (7.51) explicitly), and they are also the same as the components of an embedding tensor in Ref. [32].

Now let us show that the 3-bracket (7.50) is indeed equivalent to Bagger and Lambert's 3-bracket [38]. Writing $X = X_{k\hat{k}} t^{\hat{k}k}$, and $\bar{Z} = \bar{Z}^{\hat{m}m} \bar{t}_{m\hat{m}}$, by Eq. (7.50), one can get

$$[X, Y; \bar{Z}] = k(X\bar{Z}Y - Y\bar{Z}X)_{n\hat{n}} t^{\hat{n}n}. \tag{7.52}$$

The right hand side is the ordinary matrix multiplication. It is exactly the same as eqn. (53) of Ref. [38]. In accordance with eq. (7.51), the gauge fields can be decomposed as

$$\begin{aligned} \tilde{A}_\mu{}^{\hat{k}k}{}_{n\hat{n}} &= A_\mu{}^{\hat{m}m}{}_{l\hat{l}} f^{\hat{k}k,\hat{l}l}{}_{m\hat{m},n\hat{n}} \\ &= A_\mu{}^{\hat{k}l}{}_{l\hat{n}} \delta^k{}_n - A_\mu{}^{\hat{l}k}{}_{n\hat{l}} \delta^{\hat{k}}{}_{\hat{n}} \\ &\equiv \hat{A}_\mu{}^{\hat{k}}{}_{\hat{n}} \delta^k{}_n + A_\mu{}^k{}_n \delta^{\hat{k}}{}_{\hat{n}}. \end{aligned} \tag{7.53}$$

So the 3-bracket (7.52) and the FI (7.19) generate a $U(M) \times U(N)$ gauge group [38], with $\hat{A}_\mu{}^{\hat{k}}{}_{\hat{n}}$ the $U(M)$ part and $A_\mu{}^k{}_n$ the $U(N)$ part of the gauge potential. The Lie algebra of the gauge group is the bosonic part of the super Lie algebra $U(M|N)$. Indeed, the super Lie algebra $U(M|N)$ can be used to realize the 3-algebra used in this subsection (see section 8.1).

The supersymmetry transformation law and the Lagrangian in this theory can be obtained by substituting the expression (7.51) of the structure constants into Eqs. (7.22) and (7.21), and replacing $A_\mu{}^b{}_a$ by $\frac{1}{k} A_\mu{}^b{}_a$. To make the paper self-contained, we include the results in Appendix C.2.2. The SUSY transformation law (C.27) and the Lagrangian (C.24) are in agreement with the $D = 3, \mathcal{N} = 6$ $U(M) \times U(N)$ CSM theory, which has been derived from the ordinary Lie algebra approach in Ref. [35] and from the 3-algebra approach in Ref. [38].

This theory is conjectured to be the dual gauge theory of M2-branes a $\mathbf{C}^4/\mathbf{Z}_k$ singularity. If $M = N$, this theory becomes the well-known ABJM theory [25, 29, 30].

# CHAPTER 8

# $\mathcal{N}$=6, 8 THEORIES IN TERMS OF THE BOSONIC PARTS OF SUPERALGEBRAS

In this chapter, we first try to find a super Lie algebra which can be used to realize the hermitian 3-algebra and the Nambu 3-algebra. We then derive the ordinary Lie algebra constructions of the $\mathcal{N} = 6, 8$ theories by using the super-Lie-algebra realization of 3-algebras.

## 8.1 $\mathcal{N} = 6$ Theories in Terms of the Bosonic Parts of Superalgebras

In this section, we derive the super Lie algebra used to realize the hermitian 3-algebra and the Nambu 3-algebra by decomposing the super Lie algebra (5.2), and classify the gauge groups of the $\mathcal{N} = 6$ theories. For convenience, we cite the super Lie algebra (5.2) used to realize the symplectic 3-algebra here:

$$\begin{aligned} &[M^m, M^n] = C^{mn}{}_s M^s, \\ &[M^m, Q_I] = -\tau^m_{IJ}\omega^{JK} Q_K, \\ &\{Q_I, Q_J\} = \tau^m_{IJ} k_{mn} M^n. \end{aligned} \tag{8.1}$$

If we use the fermionic generators $Q_I$ of (8.1) to realize the 3-algebra generators $T_I$ in Eq. (7.13), i.e., $T_I \doteq Q_I$, then Eq. (7.13) becomes

$$Q_I = Q_{a\alpha} = \omega_{a\alpha,b\beta} Q^{b\beta} = Q^{a2}\delta_{1\alpha} - Q^{a1}\delta_{2\alpha}. \tag{8.2}$$

Recall that $Q_I$ furnish a pseudo-real (quaternion) representation of the bosonic part of the super Lie algebra (8.1), and we decompose this pseudo-real representation into a complex representation and its complex conjugate representation for promoting the $\mathcal{N} = 5$ supersymmetry to $\mathcal{N} = 6$. So, with the decomposition (8.2), if the fermionic generators $Q^{a1}$ furnish a complex representation of the bosonic part of (8.1), then $Q^{a2}$

must furnish a complex conjugate representation of the bosonic part of (8.1). Namely, if we define

$$Q^a = Q^{a1} \quad \text{and} \quad \bar{Q}_a = Q^{a2}, \tag{8.3}$$

we must have

$$[M^m, Q^a] = -\tau^{ma}{}_b Q^b \quad \text{and} \quad [M^m, \bar{Q}_a] = \tau^{mb}{}_a \bar{Q}_b, \tag{8.4}$$

where $\tau^{ma}{}_b$ are anti-hermitian, i.e.,

$$\tau^{*ma}{}_b = -\tau^{mb}{}_a. \tag{8.5}$$

Substituting

$$Q_I = \bar{Q}_a \delta_{1\alpha} - Q^a \delta_{2\alpha} \tag{8.6}$$

into the LHS of the second equation of (8.1) gives

$$[M^m, \bar{Q}_a \delta_{1\alpha} - Q^a \delta_{2\alpha}] = \tau^{mb}{}_a \bar{Q}_b \delta_{1\alpha} + \tau^{ma}{}_b Q^b_{2\alpha}. \tag{8.7}$$

Comparing the RHS with the RHS of the second equation of (8.1), we obtain

$$\tau^{mJ}{}_I = \tau^{mb}{}_a \delta_{1\alpha}\delta_{1\beta} - \tau^{ma}{}_b \delta_{2\alpha}\delta_{2\beta}. \tag{8.8}$$

By (8.5), the RHS is a direct sum of $\tau^{mb}{}_a$ and its complex conjugate. So the pseudo-real representation is indeed decomposed into a complex representation and its complex conjugate representation. Substituting (8.6) and (8.8) into the LHS and RHS of the third equation of (8.1), respectively, we obtain

$$\{\bar{Q}_a \delta_{1\alpha} - Q^a \delta_{2\alpha}, \bar{Q}_b \delta_{1\beta} - Q^b \delta_{2\beta}\} = -(\tau^{mb}{}_a k_{mn} M^n \delta_{1\alpha}\delta_{2\beta} + \tau^{ma}{}_b k_{mn} M^n \delta_{2\alpha}\delta_{1\beta}), \tag{8.9}$$

where we have used Eq. (7.5). The anticommutators can be easily read off from the above equation:

$$\{Q^b, \bar{Q}_a\} = \tau^{mb}{}_a k_{mn} M^n, \quad \{\bar{Q}_a, \bar{Q}_b\} = \{Q^a, Q^b\} = 0. \tag{8.10}$$

In summary, the super Lie algebra used to realize the $\mathcal{N} = 6$ (hermitian) 3-algebra is the following:

$$\begin{aligned}
&[M^m, M^n] = C^{mn}{}_s M^s, \\
&[M^m, Q^a] = -\tau^{ma}{}_b Q^b, \quad [M^m, \bar{Q}_a] = \tau^{mb}{}_a \bar{Q}_b, \\
&\{Q^a, \bar{Q}_b\} = \tau^{ma}{}_b k_{mn} M^n, \quad \{\bar{Q}_a, \bar{Q}_b\} = \{Q^a, Q^b\} = 0.
\end{aligned} \tag{8.11}$$

In this way, we rederive the above super Lie algebra by decomposing the super Lie algebra (8.1) properly. The super Lie algebras $OSp(2|2N)$ and $U(M|N)$ (or its cousins $SU(M|N)$ and $PSU(M|N)$) take the form of (8.11).

With these decompositions, the double graded commutator (see section 5.1)

$$[\{Q_I, Q_J\}, Q_K] = k_{mn}\tau^m_{IJ}\tau^{n\,L}_K Q_L \tag{8.12}$$

is decomposed into two sets:

$$[\{Q^b, \bar{Q}_a\}, Q^c] = -k_{mn}\tau^{mb}{}_a\tau^{nc}{}_d Q^d, \quad [\{Q^a, \bar{Q}_b\}, \bar{Q}_c] = k_{mn}\tau^{ma}{}_b\tau^{nd}{}_c\bar{Q}_d. \tag{8.13}$$

However, their structure constants are related by a reality condition (see Eqs. (8.16) and (8.21)). So we need only to consider the first equation. Recall that we use the the double commutator to realize the symplectic 3-bracket, i.e., $[T_I, T_J; T_K] \doteq [\{Q_I, Q_J\}, Q_K]$. Comparing the decomposition of (8.12) with (7.17), we are led to the following equations:

$$[t^b, t^c; \bar{t}_a] \doteq [\{Q^b, \bar{Q}_a\}, Q^c] = -k_{mn}\tau^{mb}{}_a\tau^{nc}{}_d Q^d, \tag{8.14}$$

$$[t^b, t^c; \bar{t}_a]^* \doteq -[\{Q^a, \bar{Q}_b\}, \bar{Q}_c] = -k_{mn}\tau^{ma}{}_b\tau^{nd}{}_c\bar{Q}_d. \tag{8.15}$$

where the LHS of the first equation is the 3-bracket of the hermitian 3-algebra, and $t^a$ are the generators of the hermitian 3-algebra (see section 7.1). The structure constants can be read off immediately:

$$f^{bc}{}_{ad} = -k_{mn}\tau^{mb}{}_a\tau^{nc}{}_d. \tag{8.16}$$

It is straightforward to verify that the above tensor product is a solution of the FI (7.19) of the hermitian 3-algebra (for convenience, we cite it here):

$$f^{fc}{}_{dg}f^{ag}{}_{eb} - f^{af}{}_{gb}f^{gc}{}_{de} + f^{cf}{}_{eg}f^{ag}{}_{db} - f^{ac}{}_{gb}f^{gf}{}_{ed} = 0. \tag{8.17}$$

The solution (8.16) is first discovered by BL [38], using a different approach. Similarly, the $Q_IQ_JQ_K$ Jacobi identity is decomposed into two sets: the $Q^bQ^c\bar{Q}_a$ Jacobi identity and the $\bar{Q}_b\bar{Q}_cQ^a$ Jacobi identity. Let us examine the $Q^bQ^c\bar{Q}_a$ Jacobi identity:

$$[\{Q^b, \bar{Q}_a\}, Q^c] + [\{\bar{Q}_a\, Q^c\}, Q^b] + [\{Q^c, Q^b\}, \bar{Q}_a] = 0. \tag{8.18}$$

By $\{Q^c, Q^b\} = 0$, the last term of the LHS vanishes. The equation for the remaining two terms implies that

$$k_{mn}\tau^{mb}{}_a\tau^{nc}{}_d + k_{mn}\tau^{mc}{}_a\tau^{nb}{}_d = 0. \tag{8.19}$$

Namely, the structure constants $f^{bc}{}_{ad}$ are antisymmetric in the first two indices and in the last two indices:

$$f^{bc}{}_{ad} = -f^{cb}{}_{ad} = f^{cb}{}_{da}. \tag{8.20}$$

Also, the reality condition (8.5) implies that the structure constants satisfy the reality condition:

$$f^{*ab}{}_{cd} = f^{cd}{}_{ab}. \tag{8.21}$$

Eqs (8.20) and (8.21) are nothing but Eqs (7.20) and (7.12), respectively.

Here we would like to demonstrate that the FI (8.17), satisfied by the structure constants, is equivalent to the $MMQ$ or $MM\bar{Q}$ Jacobi identity of the super Lie algebra (8.11). With Eq. (8.6), the FI (5.6) is decomposed into eight sets; one of them reads

$$\begin{aligned} & [\{\bar{Q}_a, Q^b\}, [\{\bar{Q}_e, Q^f\}, Q^c]] \qquad\qquad (8.22)\\ = \; & [\{[\{\bar{Q}_a, Q^b\}\bar{Q}_e], Q^f\}, Q^c] + [\{\bar{Q}_e, [\{\bar{Q}_a, Q^b\}, Q^f]\}, Q^c] + [\{\bar{Q}_e, Q^f\}, [\{\bar{Q}_a, Q^b\}, Q^c]]. \end{aligned}$$

Substituting (8.13) into this equation shows that it precisely coincides with the FI (8.17). The rest (seven) sets can be also converted into the FI (8.17). So it is sufficient to examine Eq. (8.22). On the other hand, by using the super Lie algebra (8.11), one can convert Eq. (8.22) into the following equation:

$$\tau^{mb}{}_a \tau^{nf}{}_e([M_n, [M_m, Q^c]] - [M_m, [M_n, Q^c]] + [[M_m, M_n], Q^c]) = 0, \tag{8.23}$$

which is the $MMQ$ Jacobi identity of the super Lie algebra (8.11). One can also derive the above equation by decomposing Eq. (5.7). With $Q^c$ replaced by $\bar{Q}_c$, Eq. (8.22) becomes another set FI decomposed from (5.6). It can be converted into $MM\bar{Q}$ Jacobi identity of (8.11). Therefore, the FI (8.17) is indeed equivalent to the $MMQ$ or $MM\bar{Q}$ Jacobi identity of the super Lie algebra (8.11).

For a more mathematical approach, see Ref. [37, 42, 46], in which the relations between the hermitian 3-algebras and Lie superalgebras are discussed by using Lie algebra representation theories.

Substituting Eq. (8.16) into the Lagrangian (7.21) and the SUSY law (7.22) gives the ordinary Lie algebra constructions of the $\mathcal{N} = 6$ theories. The bosonic parts of the super Lie algebras $OSp(2|2N)$ and $U(M|N)$ (or its cousins $SU(M|N)$ and $PSU(M|N)$) can be selected as the Lie algebras of the gauge groups of the $\mathcal{N} = 6$ theories (see (8.11)). In particular, if the super Lie algebra is $PSU(2|2)$, then (8.14) becomes the Nambu bracket, and the $\mathcal{N} = 6$ supersymmetry gets enhanced to $\mathcal{N} = 8$. This will be the topic of the next section.

## 8.2 $\mathcal{N}=8$ Theory in Terms of the Bosonic Part of $PSU(2|2)$

In section (7.2), we have realized the Nambu 3-algebra in terms of a set of $SU(2)\times SU(2)$ $\sigma$-matrices. In this section, we show explicitly that the Nambu 3-algebra can be realized in term of $PSU(2|2)$. Some useful (anti-) commutators of $PSU(2|2)$ generators are

$$\begin{aligned}
&[M_\alpha{}^\gamma, Q_\beta{}^{\dot\beta}] = \delta_\beta{}^\gamma Q_\alpha{}^{\dot\beta} - \frac{1}{2}\delta_\alpha{}^\gamma Q_\beta{}^{\dot\beta},\\
&[M_{\dot\delta}{}^{\dot\beta}, Q_\alpha{}^{\dot\gamma}] = -\delta_{\dot\delta}{}^{\dot\gamma} Q_\alpha{}^{\dot\beta} + \frac{1}{2}\delta_{\dot\delta}{}^{\dot\beta} Q_\alpha{}^{\dot\gamma},\\
&\{Q_\alpha{}^{\dot\alpha}, \bar Q_{\dot\beta}{}^\beta\} = \delta_{\dot\beta}{}^{\dot\alpha} M_\alpha{}^\beta + \delta_\alpha{}^\beta M_{\dot\beta}{}^{\dot\alpha},\\
&\{Q_\alpha{}^{\dot\alpha}, Q_\beta{}^{\dot\beta}\} = 0,
\end{aligned} \tag{8.24}$$

where $\alpha,\dot\beta = 1,2$ are $SU(2)\times SU(2)$ indices. We use the antisymmetric matrix $\epsilon_{\alpha\beta}$ ($\epsilon_{\dot\alpha\dot\beta}$) to lower undotted (dotted) indices. Define the $SU(2)\times SU(2)$ $\sigma$-matrices as (see section 7.2):

$$\begin{aligned}
&\sigma^a{}_\alpha{}^{\dot\alpha} = (\sigma^1,\sigma^2,\sigma^3, i\mathbb{I}), &&\sigma^{a\dagger}{}_{\dot\alpha}{}^\alpha = (\sigma^1,\sigma^2,\sigma^3,-i\mathbb{I})\\
&\sigma^{ab}{}_\alpha{}^\beta = \frac{1}{4}(\sigma^a\sigma^{b\dagger} - \sigma^b\sigma^{a\dagger})_\alpha{}^\beta, &&\bar\sigma^{ab}{}_{\dot\alpha}{}^{\dot\beta} = \frac{1}{4}(\sigma^{a\dagger}\sigma^b - \sigma^{b\dagger}\sigma^a)_{\dot\alpha}{}^{\dot\beta},
\end{aligned} \tag{8.25}$$

where $\sigma^{ab}$ and $\bar\sigma^{ab}$ satisfy the further 'duality' conditions

$$\sigma^{ab} = -\frac{1}{2}\varepsilon^{abcd}\sigma_{cd}, \qquad \bar\sigma^{ab} = \frac{1}{2}\varepsilon^{abcd}\bar\sigma_{cd}. \tag{8.26}$$

Since we wish to work in the vector representation of $SO(4)$, it is useful to define

$$\begin{aligned}
&Q^a = \frac{1}{2}\sigma^{a\dagger}{}_{\dot\alpha}{}^\alpha Q_\alpha{}^{\dot\alpha}, \quad \bar Q^a = \frac{1}{2}\sigma^a{}_\alpha{}^{\dot\alpha}\bar Q_{\dot\alpha}{}^\alpha, \quad M^{ab} = -(\sigma^{ab}{}_\alpha{}^\beta M_\beta{}^\alpha + \bar\sigma^{ab}{}_{\dot\alpha}{}^{\dot\beta} M_{\dot\beta}{}^{\dot\alpha})\\
&M_\alpha{}^\beta = M_{ab}\sigma^{ab}{}_\alpha{}^\beta, \quad M_{\dot\alpha}{}^{\dot\beta} = M_{ab}\bar\sigma^{ab}{}_{\dot\alpha}{}^{\dot\beta}.
\end{aligned} \tag{8.27}$$

After some work, we obtain

$$[t^b, t^c; \bar t^a] \doteq [\{Q^b, \bar Q^a\}, Q^c] = \frac{1}{2}\varepsilon^{bcad} Q_d. \tag{8.28}$$

Namely, the double graded commutator is indeed a realization of the Nambu 3-bracket. Also, the FI satisfied by the Nambu 3-bracket is equivalent to the $MMQ$ Jacobi identity of (8.24), as we proved in the last section. Therefore the Nambu 3-algebra is realized in terms of the super Lie algebra $PSU(2|2)$. Hence the bosonic part of $PSU(2|2)$, $SU(2)\times SU(2)\cong SO(4)$, is the Lie algebra of the gauge group of the $\mathcal{N}=8$ BLG theory. And the matter fields are in the bifundamental representation of $SU(2) \times SU(2)$ or the vector representation of $SO(4)$. The same theory is obtained in Ref. [34] by promoting the $\mathcal{N} = 4$ supersymmetry to $\mathcal{N} = 8$.

Eq. (8.28) may be counterintuitive at first sigh, since the anticommutator satisfies $\{Q^b, \bar{Q}^a\} = \{\bar{Q}^a, Q^b\}$, i.e., it seems that it is symmetric in $ab$. However, there is no clash with fact that (8.28) is antisymmetric in $ab$ if we notice that

$$\{Q^b, \bar{Q}^a\} = \frac{1}{4}\varepsilon^{bacd} M_{cd} = -\{Q^a, \bar{Q}^b\} = \{\bar{Q}^a, Q^b\}, \tag{8.29}$$

namely, the last two anticommutators are *different*.

It is well known that the Nambu 3-bracket is difficult to quantize [12, 13]. However, if we promote the fermionic and bosonic generators of (8.24) as quantum mechanical operators, and promote (8.28) as a quantum mechanical double graded commutator, our approach may provide a quantization scheme for the Nambu 3-bracket. Similarly, the 3-brackets of the symplectic and hermitian 3-algebras may be quantized in the same fashion.

# CHAPTER 9

# CONCLUSIONS

In this thesis, we have combined the symplectic 3-algebra with the superspace formalism by letting the matter superfields take values in the symplectic 3-algebra. Based on the 3-algebra, we then have constructed the general $\mathcal{N}=5$ CMS theory by enhancing the $\mathcal{N}=1$ supersymmetry to $\mathcal{N}=5$. The $\mathcal{N}=5$ Lagrangian is same as the one derived with an on-shell approach [40].

We have constructed the general $\mathcal{N}=4$ CSM theory by decomposing one $\mathcal{N}=5$ multiplet into a $\mathcal{N}=4$ untwisted hypermultiplet and a $\mathcal{N}=4$ twisted hypermultiplet, and then proposing a new superpotential. In deriving the general $\mathcal{N}=4$ CSM theory, we have also decomposed the set of 3-algebra generators into two sets of 3-algebra generators. As a result, both the FIs and 3-brackets are decomposed into 4 sets. The resulting general $\mathcal{N}=4$ CSM theory is a quiver gauge theory based on the 3-algebra. We have also examined the closure of the $\mathcal{N}=4$ algebra.

We then have realized the symplectic 3-algebra in terms of the super Lie algebra (5.2). The 3-bracket is realized in terms of a double graded bracket: $[T_I, T_J; T_K] \doteq [\{Q_I, Q_J\}, Q_K]$, where $Q_I$ are the fermionic generators; the structure constants of the 3-algebra are just the structure constants of the double graded bracket, i.e., $f_{IJKL} = k_{mn}\tau^m_{IJ}\tau^n_{KL}$. The fundamental identity of the 3-algebra is equivalent to the $MMQ$ Jacobi identity of the super Lie algebra, where $M$s are the bosonic generators in the super Lie algebra. The linear constraint equation $f_{(IJK)L} = 0$, required by the enhancement of the supersymmetry, is equivalent to the $QQQ$ Jacobi identity.

We have constructed a new super Lie algebra by requiring that the bosonic subalgebras of two simple super Lie algebras share one simple factor. The bosonic part of this new super algebra can be selected as the Lie algebra of the $\mathcal{N}=4$ quiver gauge theory. We have demonstrated how to 'fuse' two simple super Lie algebras into single one, by constructing an explicit example of this new super Lie algebra.

We have also analyzed the relations between the symplectic 3-algebra and the ordinary Lie algebra. The fundamental identity of 3-algebra can be solved in terms of a tensor product: $f_{IJKL} = k_{mn}\tau^m_{IJ}\tau^n_{KL}$. We have proved that the structure constants $f_{IJKL}$ furnish a quaternion representation of the bosonic part of the super Lie algebra (5.2), and $f_{IJKL}$ also play a role of Killing-Cartan metric. We found that the FI of the 3-algebra can be converted into an ordinary commutator (5.19); the structure constants of the commutator are (5.21). The FI of the 3-algebra can be understood as the statement that the structure constants of the commutator (5.21) are total antisymmetric (see Eqs. (5.22)).

We have proved that the components of an embedding tensor [31, 32], used to construct the $D = 3$ extended supergravity theories, are just the structure constants of the 3-algebra. Hence the concepts and techniques of the 3-algebra may be used to construct new $D = 3$ extended supergravity theories.

We have succeeded in enhancing the $\mathcal{N} = 5$ supersymmetry to $\mathcal{N} = 6$ by decomposing the sympelctic 3-algebra and the fields properly. At the same time, we also demonstrate that the FI and the symmetry and reality properties of the structure constants of the $\mathcal{N} = 6$ hermitian 3-algebra can be derived from the $\mathcal{N} = 5$ (symplectic) counterparts. In the particular case of $f^{abcd} \propto \varepsilon^{abcd}$, the $\mathcal{N} = 6$ supersymmetry is promoted to $\mathcal{N} = 8$, hence the $\mathcal{N} = 6$ theory becomes the $\mathcal{N} = 8$ BLG theory.

We have shown that the ($\mathcal{N} = 6$) hermitian 3-algebra and the Nambu algebra can be also realized in terms of super Lie algebras, and introduced a scheme for quantizing the 3-brackets.

The general $\mathcal{N} = 5, 6, 8$ CSM theories in terms of ordinary Lie algebras are rederived. We have been able to derive all known $\mathcal{N} = 4, 5, 6, 8$ superconformal Chern-Simons matter theories, as well as some new $\mathcal{N} = 4$ quiver gauge theories. Thus our superspace formulation for the super-Lie-algebra realization of symplectic 3-algebras provides a unified framework of all known $\mathcal{N} = 4, 5, 6, 8$ CSM theories, including new examples of $\mathcal{N} = 4$ quiver gauge theories as well. It would be nice to investigate the physical significance of this unified framework.

It would be nice to redrive them by brane constructions, and to find out their gravity duals. (Most of them are not found yet.) It would also be very interesting to investigate the integrability of these theories.

The 'fused' super Lie algebra might be independently interesting in its own right. It would be nice to investigate its structure in detail.

The 'meshy' quiver diagram (6.26) is just a special example. It would be nice to work out the most general structure of the 'meshy' quiver diagrams.

# APPENDIX A

# CONVENTIONS AND USEFUL IDENTITIES

## A.1 Spinor Algebra

In $1+2$ dimensions, the gamma matrices are defined as

$$(\gamma_\mu)_\alpha{}^\gamma(\gamma_\nu)_\gamma{}^\beta + (\gamma_\nu)_\alpha{}^\gamma(\gamma_\mu)_\gamma{}^\beta = 2\eta_{\mu\nu}\delta_\alpha{}^\beta. \tag{A.1}$$

For the metric we use the $(-,+,+)$ convention. The gamma matrices in the Majorana representation can be defined in terms of Pauli matrices: $(\gamma_\mu)_\alpha{}^\beta = (i\sigma_2, \sigma_1, \sigma_3)$, satisfying the important identity

$$(\gamma_\mu)_\alpha{}^\gamma(\gamma_\nu)_\gamma{}^\beta = \eta_{\mu\nu}\delta_\alpha{}^\beta + \varepsilon_{\mu\nu\lambda}(\gamma^\lambda)_\alpha{}^\beta. \tag{A.2}$$

We also define $\varepsilon^{\mu\nu\lambda} = -\varepsilon_{\mu\nu\lambda}$. So $\varepsilon_{\mu\nu\lambda}\varepsilon^{\rho\nu\lambda} = -2\delta_\mu{}^\rho$. We raise and lower spinor indices with an antisymmetric matrix $\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta}$, with $\epsilon_{12} = -1$. For example, $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$ and $\gamma^\mu_{\alpha\beta} = \epsilon_{\beta\gamma}(\gamma^\mu)_\alpha{}^\gamma$, where $\psi_\beta$ is a Majorana spinor. Notice that $\gamma^\mu_{\alpha\beta} = (\mathbb{I}, -\sigma^3, \sigma^1)$ are symmetric in $\alpha\beta$. A vector can be represented by a symmetric bispinor and vice versa:

$$A_{\alpha\beta} = A_\mu\gamma^\mu_{\alpha\beta}, \qquad A_\mu = -\frac{1}{2}\gamma_\mu^{\alpha\beta}A_{\alpha\beta}. \tag{A.3}$$

We use the following spinor summation convention:

$$\psi\chi = \psi^\alpha\chi_\alpha, \qquad \psi\gamma_\mu\chi = \psi^\alpha(\gamma_\mu)_\alpha{}^\beta\chi_\beta, \tag{A.4}$$

where $\psi$ and $\chi$ are anticommuting Majorana spinors. In $1+2$ dimensions the Fierz transformations are

$$\begin{aligned}
(\lambda\chi)\psi &= -\frac{1}{2}(\lambda\psi)\chi - \frac{1}{2}(\lambda\gamma_\nu\psi)\gamma^\nu\chi, \qquad\qquad (\text{A.5})\\
(\psi_1\psi_2)(\psi_3\psi_4) &= (\psi_1\psi_2)(\psi_4\psi_3) = -\frac{1}{2}(\psi_1\psi_3)(\psi_4\psi_2) - \frac{1}{2}(\psi_1\gamma_\nu\psi_3)(\psi_4\gamma^\nu\psi_2),\\
(\psi_1\gamma_\mu\psi_2)(\psi_3\psi_4) &= -\frac{1}{2}(\psi_1\gamma_\mu\psi_3)(\psi_4\psi_2) - \frac{1}{2}(\psi_1\psi_3)(\psi_4\gamma_\mu\psi_2) + \frac{1}{2}\varepsilon_{\mu\nu\lambda}(\psi_1\gamma^\nu\psi_3)(\psi_4\gamma^\lambda\psi_2).
\end{aligned}$$

## A.2 The $\mathcal{N} = 1$ Superspace

In this subsection, we mainly follow the conventions of Ref. [35]. We denote the superspace coordinates as $\theta^\alpha$. A real scalar superfield $\Phi$ can be expanded as

$$\Phi = \phi + i\theta\psi - \frac{i}{2}\theta^2 F, \tag{A.6}$$

where $\theta$ and $\psi$ are Majorana spinors. The superalgebra

$$\{Q_\alpha, Q_\beta\} = -2\gamma^\mu_{\alpha\beta} P_\mu \tag{A.7}$$

can be realized in terms of superspace derivatives:

$$Q_\alpha = i\partial_\alpha + \theta^\beta \partial_{\beta\alpha}. \tag{A.8}$$

The supercovariant derivative must anticommute with $Q_\alpha$; it takes the following form:

$$\mathscr{D}_\alpha = \partial_\alpha + i\theta^\beta \partial_{\beta\alpha}. \tag{A.9}$$

The supersymmetry transformation of $\Phi$ is defined as

$$\delta\Phi = -i\epsilon^\alpha Q_\alpha \Phi \equiv \delta\phi + i\theta\delta\psi - \frac{i}{2}\theta^2 \delta F. \tag{A.10}$$

Equating powers of $\theta^\alpha$ gives the supersymmetry transformations of the component fields:

$$\delta\phi = i\epsilon^\alpha \psi_\alpha, \tag{A.11}$$
$$\delta\psi_\alpha = -\partial_\alpha{}^\beta \phi \epsilon_\beta - F\epsilon_\alpha, \tag{A.12}$$
$$\delta F = i\epsilon^\alpha \partial_\alpha{}^\beta \psi_\beta. \tag{A.13}$$

In the Wess-Zumino gauge, the superconnection becomes

$$\Gamma_\alpha = i\theta^\beta A_{\alpha\beta} + \theta^2 \chi_\alpha, \tag{A.14}$$

and the supersymmetry transformations for the component fields are

$$\delta A_\mu = -i\epsilon^\alpha (\gamma_\mu)_\alpha{}^\beta \chi_\beta, \tag{A.15}$$
$$\delta\chi_\alpha = -\frac{1}{2} F_{\mu\nu} (\gamma^{\mu\nu})_\alpha{}^\beta \epsilon_\beta. \tag{A.16}$$

The Berezin integral is defined as

$$\int d^2\theta \theta^2 = -4. \tag{A.17}$$

The superpotential is given by

$$\mathcal{L}_W = \frac{i}{2} \int d^2\theta W(\Phi) = -\frac{i}{2} W''(\phi)\psi^2 - W'(\phi) F. \tag{A.18}$$

## A.3 $SU(2) \times SU(2)$ Identities

We define the 4 sigma matrices as

$$\sigma^a{}_A{}^{\dot{B}} = (\sigma^1, \sigma^2, \sigma^3, i\mathbb{I}), \tag{A.19}$$

by which one can establish a connection between the $SU(2) \times SU(2)$ and $SO(4)$ group. These sigma matrices satisfy the following Clifford algebra:

$$\sigma^a{}_A{}^{\dot{C}} \sigma^{b\dagger}{}_{\dot{C}}{}^B + \sigma^b{}_A{}^{\dot{C}} \sigma^{a\dagger}{}_{\dot{C}}{}^B = 2\delta^{ab}\delta_A{}^B, \tag{A.20}$$

$$\sigma^{a\dagger}{}_{\dot{A}}{}^C \sigma^b{}_C{}^{\dot{B}} + \sigma^{b\dagger}{}_{\dot{A}}{}^C \sigma^a{}_C{}^{\dot{B}} = 2\delta^{ab}\delta_{\dot{A}}{}^{\dot{B}}. \tag{A.21}$$

We use antisymmetric matrices

$$\epsilon_{AB} = -\epsilon^{AB} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon_{\dot{A}\dot{B}} = -\epsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.22}$$

to raise or lower undotted and dotted indices, respectively. For example, $\sigma^{a\dagger \dot{A}B} = \epsilon^{\dot{A}\dot{B}} \sigma^{a\dagger}{}_{\dot{B}}{}^B$ and $\sigma^{aB\dot{A}} = \epsilon^{BC}\sigma^a{}_C{}^{\dot{A}}$. The sigma matrix $\sigma^a$ satisfies a reality condition

$$\sigma^{a\dagger}{}_{\dot{A}}{}^B = -\epsilon^{BC}\epsilon_{\dot{A}\dot{B}}\sigma^a{}_C{}^{\dot{B}}, \quad \text{or} \quad \sigma^{a\dagger\dot{A}B} = -\sigma^{aB\dot{A}}. \tag{A.23}$$

The antisymmetric matrix $\epsilon_{AB}$ satisfies an important identity

$$\epsilon_{AB}\epsilon^{CD} = -(\delta_A{}^C\delta_B{}^D - \delta_A{}^D\delta_B{}^C), \tag{A.24}$$

and $\epsilon_{\dot{A}\dot{B}}$ satisfies a similar identity.

Define

$$\sigma^{A\dot{B}} \equiv c_a\sigma^{aA\dot{B}} \quad \text{and} \quad c_a c^a = 1, \tag{A.25}$$

where $c_a$ are real coefficients, then the following identity holds

$$\sigma^{A\dot{C}}\sigma^{B\dot{D}} - \sigma^{A\dot{D}}\sigma^{B\dot{C}} = \epsilon^{AB}\epsilon^{\dot{C}\dot{D}}. \tag{A.26}$$

This identity is useful when we construct the $\mathcal{N} = 4$ theory. Define the parameter for the $\mathcal{N} = 4$ supersymmetry transformations as $\epsilon^{A\dot{B}} = \epsilon_a\sigma^{aA\dot{B}}$. The following identities are useful in checking the closure of the $\mathcal{N} = 4$ superalgebra:

$$\begin{aligned}
i(\epsilon_1^{A\dot{C}}\epsilon^{\dagger}_{2\dot{C}}{}^{B} - \epsilon_2^{A\dot{C}}\epsilon^{\dagger}_{1\dot{C}}{}^{B}) &\equiv u^{AB} = u^{BA}, \\
i(\epsilon_1^{\dagger\dot{A}C}\epsilon_{2C}{}^{\dot{B}} - \epsilon_2^{\dagger\dot{A}C}\epsilon_{1C}{}^{\dot{B}}) &\equiv u^{\dot{A}\dot{B}} = u^{\dot{B}\dot{A}}, \\
i(\epsilon_{1A}{}^{\dot{A}}\gamma^{\mu}\epsilon^{\dagger}_{2\dot{A}}{}^{B} - \epsilon_{2A}{}^{\dot{A}}\gamma^{\mu}\epsilon^{\dagger}_{1\dot{A}}{}^{B}) &= i\epsilon_1^{C\dot{C}}\gamma^{\mu}\epsilon_{2C\dot{C}}\delta_A{}^B \equiv v^{\mu}\delta_A{}^B, \\
2(\epsilon_{1A\dot{A}}\epsilon^{\dagger}_{2\dot{B}B} - \epsilon_{2A\dot{A}}\epsilon^{\dagger}_{1\dot{B}B}) &= (\epsilon_{1A}{}^{\dot{C}}\epsilon^{\dagger}_{2\dot{C}B} - \epsilon_{2A}{}^{\dot{C}}\epsilon^{\dagger}_{1\dot{C}B})\epsilon_{\dot{A}\dot{B}} \\
&\quad +(\epsilon^{\dagger}_{1\dot{B}}{}^{C}\epsilon_{2C\dot{A}} - \epsilon^{\dagger}_{2\dot{B}}{}^{C}\epsilon_{1C\dot{A}})\epsilon_{AB}, \\
i\epsilon_{AB}\epsilon_{\dot{C}\dot{D}}\epsilon_1^{E\dot{E}}\gamma^{\mu}\epsilon_{2E\dot{E}} &= i(\epsilon_{1B\dot{C}}\gamma^{\mu}\epsilon^{\dagger}_{2\dot{D}A} - \epsilon_{2B\dot{C}}\gamma^{\mu}\epsilon^{\dagger}_{1\dot{D}A}) \\
&\quad -i(\epsilon_{1A\dot{C}}\gamma^{\mu}\epsilon^{\dagger}_{2\dot{D}B} - \epsilon_{2A\dot{C}}\gamma^{\mu}\epsilon^{\dagger}_{1\dot{D}B}).
\end{aligned} \tag{A.27}$$

## A.4 $Sp(4) \cong SO(5)$ Identities

In this subsection, in order to avoid introducing too many indices into the theory, we still use the capital letters $A, B, \cdots$ to label the $Sp(4)$ indices. However, now the index $A$ runs from 1 to 4. (In section A.3, the indices $A$ and $\dot{B}$ run from 1 to 2.) We hope this does not cause any confusion.

Since $Sp(4) \cong SO(5)$, it is useful to introduce the $SO(5)$ gamma matrices. We define the $SO(5)$ gamma matrices as

$$\gamma^{a\,B}_{A} = \begin{pmatrix} 0 & \sigma^a \\ \sigma^{a\dagger} & 0 \end{pmatrix}, \qquad \gamma^{5\,B}_{A} = (\gamma^1\gamma^2\gamma^3\gamma^4)_A{}^B, \tag{A.28}$$

where $\sigma^a$ are defined by (A.19). Notice that $\gamma^{mB}_A$ $(m = 1, \cdots, 5)$ are hermitian, satisfying the Clifford algebra

$$\gamma^{mC}_{A}\gamma^{nB}_{C} + \gamma^{nC}_{A}\gamma^{mB}_{C} = 2\delta^{mn}\delta_A{}^B. \tag{A.29}$$

We use an antisymmetric matrix $\omega_{AB} = -\omega^{AB}$ to lower and raise indices; for instance

$$\gamma^{mAB} = \omega^{AC}\gamma^{mB}_{C}. \tag{A.30}$$

It can be chosen as the charge conjugate matrix:

$$\omega^{AB} = \begin{pmatrix} \epsilon^{AB} & 0 \\ 0 & \epsilon^{\dot{A}\dot{B}} \end{pmatrix}. \tag{A.31}$$

(Recall that $A$ and $\dot{B}$ of the RHS run from 1 to 2.)

By the definition (A.28) and the convention (A.30), the gamma matrix $\gamma^m$ is anti-symmetric and traceless, and satisfies a reality condition

$$\gamma^{mAB} = -\gamma^{mBA} \quad , \quad \gamma^{mA}_{A} = 0 \quad \text{and} \quad \gamma^{m*}_{AB} = \gamma^{mAB} = \omega^{AC}\omega^{BD}\gamma^{m}_{CD}. \tag{A.32}$$

The $Sp(4)$ generators are defined as

$$\Sigma_A^{mnB} = \frac{1}{4}[\gamma^m, \gamma^n]_A{}^B. \tag{A.33}$$

There are two useful $Sp(4)$ identities [35]:

$$\varepsilon^{ABCD} = -\omega^{AB}\omega^{CD} + \omega^{AC}\omega^{BD} - \omega^{AD}\omega^{BC}, \tag{A.34}$$

and

$$\begin{aligned} \varepsilon_{GABC}\varepsilon^{GDEF} &= 3!\delta^D_{[A}\delta^E_B\delta^F_{C]} \\ &= 3(-\delta^D_{[A}\omega^{EF}\omega_{BC]} + \delta^E_{[A}\omega^{DF}\omega_{BC]} - \delta^F_{[A}\omega^{DE}\omega_{BC]}). \end{aligned} \tag{A.35}$$

The following identities are useful in checking the closure of the $\mathcal{N} = 5$ superalgebra:

$$\bar{\epsilon}_1^{AC}\epsilon_{2C}{}^B - \bar{\epsilon}_2^{AC}\epsilon_{1C}{}^B = \bar{\epsilon}_1^{BC}\epsilon_{2C}{}^A - \bar{\epsilon}_2^{BC}\epsilon_{1C}{}^A \tag{A.36}$$

$$\frac{1}{2}\bar{\epsilon}_1^{CD}\gamma_\nu\epsilon_{2CD}\,\delta^A_B = \bar{\epsilon}_1^{AC}\gamma_\nu\epsilon_{2BC} - \bar{\epsilon}_2^{AC}\gamma_\nu\epsilon_{1BC} \tag{A.37}$$

$$\begin{aligned} 2\bar{\epsilon}_1^{AC}\epsilon_{2BD} - 2\bar{\epsilon}_2^{AC}\epsilon_{1BD} &= \bar{\epsilon}_1^{CE}\epsilon_{2DE}\delta^A_B - \bar{\epsilon}_2^{CE}\epsilon_{1DE}\delta^A_B \\ &- \bar{\epsilon}_1^{AE}\epsilon_{2DE}\delta^C_B + \bar{\epsilon}_2^{AE}\epsilon_{1DE}\delta^C_B \\ &+ \bar{\epsilon}_1^{AE}\epsilon_{2BE}\delta^C_D - \bar{\epsilon}_2^{AE}\epsilon_{1BE}\delta^C_D \\ &- \bar{\epsilon}_1^{CE}\epsilon_{2BE}\delta^A_D + \bar{\epsilon}_2^{CE}\epsilon_{1BE}\delta^A_D \end{aligned} \tag{A.38}$$

$$\begin{aligned} \frac{1}{2}\varepsilon_{ABCD}\bar{\epsilon}_1^{EF}\gamma_\mu\epsilon_{2EF} &= \bar{\epsilon}_{1AB}\gamma_\mu\epsilon_{2CD} - \bar{\epsilon}_{2AB}\gamma_\mu\epsilon_{1CD} \\ &+ \bar{\epsilon}_{1AD}\gamma_\mu\epsilon_{2BC} - \bar{\epsilon}_{2AD}\gamma_\mu\epsilon_{1BC} \\ &- \bar{\epsilon}_{1BD}\gamma_\mu\epsilon_{2AC} + \bar{\epsilon}_{2BD}\gamma_\mu\epsilon_{1AC} \end{aligned} \tag{A.39}$$

$$\varepsilon^{ABCD} = -\omega^{AB}\omega^{CD} + \omega^{AC}\omega^{BD} - \omega^{AD}\omega^{BC}. \tag{A.40}$$

The $Sp(4)$ indices can lowered and raised by the antisymmetric metric $\omega_{AB}$ and its inverse $\omega^{AB}$.

## A.5 $SU(4) \cong SO(6)$ Identities

Some useful $SU(4)$ identities are [38]:

$$
\begin{aligned}
\frac{1}{2}\bar{\epsilon}_1^{CD}\gamma_\nu\epsilon_{2CD}\,\delta_B^A &= \bar{\epsilon}_1^{AC}\gamma_\nu\epsilon_{2BC} - \bar{\epsilon}_2^{AC}\gamma_\nu\epsilon_{1BC} \\
2\bar{\epsilon}_1^{AC}\epsilon_{2BD} - 2\bar{\epsilon}_2^{AC}\epsilon_{1BD} &= \bar{\epsilon}_1^{CE}\epsilon_{2DE}\delta_B^A - \bar{\epsilon}_2^{CE}\epsilon_{1DE}\delta_B^A \\
&- \bar{\epsilon}_1^{AE}\epsilon_{2DE}\delta_B^C + \bar{\epsilon}_2^{AE}\epsilon_{1DE}\delta_B^C \\
&+ \bar{\epsilon}_1^{AE}\epsilon_{2BE}\delta_D^C - \bar{\epsilon}_2^{AE}\epsilon_{1BE}\delta_D^C \qquad \text{(A.41)} \\
&- \bar{\epsilon}_1^{CE}\epsilon_{2BE}\delta_D^A + \bar{\epsilon}_2^{CE}\epsilon_{1BE}\delta_D^A \\
\frac{1}{2}\varepsilon_{ABCD}\bar{\epsilon}_1^{EF}\gamma_\mu\epsilon_{2EF} &= \bar{\epsilon}_{1AB}\gamma_\mu\epsilon_{2CD} - \bar{\epsilon}_{2AB}\gamma_\mu\epsilon_{1CD} \\
&+ \bar{\epsilon}_{1AD}\gamma_\mu\epsilon_{2BC} - \bar{\epsilon}_{2AD}\gamma_\mu\epsilon_{1BC} \qquad \text{(A.42)} \\
&- \bar{\epsilon}_{1BD}\gamma_\mu\epsilon_{2AC} + \bar{\epsilon}_{2BD}\gamma_\mu\epsilon_{1AC}.
\end{aligned}
$$

# APPENDIX B

# VERIFICATION OF $SP(4)$ GLOBAL SYMMETRY OF THE $\mathcal{N}=5$ BOSONIC POTENTIAL

In this section we will prove that the bosonic potential (3.36) has an $Sp(4)$ global symmetry. For convenience, we cite it here:

$$\begin{aligned} -V &= \frac{1}{18} f_{IJKO} f^{O}{}_{LMN} (-\omega^{AC}\omega^{BE}\omega^{DF} + 2\omega^{AC}\gamma^{BE}\gamma^{DF} \\ &\quad +2\omega^{DF}\gamma^{AC}\gamma^{BE} - 4\omega^{BE}\gamma^{AC}\gamma^{DF}) Z^I_A Z^J_B Z^K_C Z^L_D Z^M_E Z^N_F . \end{aligned} \tag{B.1}$$

It can be seen that the first term is manifestly $Sp(4)$ invariant. So we need only to consider the last three terms. Denote them as $-V'$. For $-V'$, the part proportional to $Z^{(K}_{(C} Z^{L)}_{D)}$ vanishes by the FI (2.6), so the remaining part of $-V'$ is

$$\begin{aligned} -V'_A &= \frac{2}{9}(\omega^{AC}\gamma^{BE}\gamma^{DF} - \omega^{BE}\gamma^{AC}\gamma^{DF}) f_{IJKO} f^{O}{}_{LMN} Z^I_A Z^J_B Z^{[K}_{[C} Z^{L]}_{D]} Z^M_E Z^N_F \\ &\equiv \frac{2}{9}(P_1 - P_2). \end{aligned} \tag{B.2}$$

On the other hand, by using the constraint condition $f_{(IJK)O} = 0$ (see (3.31)) and the FI (2.6), one can rewrite (B.2) as

$$\begin{aligned} -V'_A &= \frac{1}{9}(\omega^{AC}\gamma^{BE}\gamma^{DF} - \omega^{BE}\gamma^{AC}\gamma^{DF} + \omega^{CD}\gamma^{AE}\gamma^{BF} - \omega^{BE}\gamma^{CD}\gamma^{AF}) \\ &\quad \times f_{IJKO} f^{O}{}_{LMN} Z^I_A Z^J_B Z^{[K}_{[C} Z^{L]}_{D]} Z^M_E Z^N_F \\ &\equiv \frac{1}{9}(P_1 - P_2 + P_3 - P_4). \end{aligned} \tag{B.3}$$

Comparing (B.2) with (B.3) gives

$$P_1 - P_2 = P_3 - P_4. \tag{B.4}$$

We observe that $2P_2 + P_4$ is an $Sp(4)$ invariant quantity:

$$\begin{aligned} 2P_2 + P_4 &= (2\omega^{BE}\gamma^{A[C}\gamma^{D]F} + \omega^{BE}\gamma^{CD}\gamma^{AF}) f_{IJKO} f^{O}{}_{LMN} Z^I_A Z^J_B Z^{[K}_{[C} Z^{L]}_{D]} Z^M_E Z^N_F \\ &= \omega^{BE}\varepsilon^{ACDF} f_{IJKO} f^{O}{}_{LMN} Z^I_A Z^J_B Z^{[K}_{[C} Z^{L]}_{D]} Z^M_E Z^N_F \\ &\equiv I. \end{aligned} \tag{B.5}$$

In the second line we have used the key identity (3.22). By using the second line of (3.22), i.e., $\varepsilon^{ABCD} = -\omega^{AB}\omega^{CD} + \omega^{AC}\omega^{BD} - \omega^{AD}\omega^{BC}$, we find that $I$ can be written as

$$-I = \varepsilon_G{}^{ACE}\varepsilon^{GBDF} f_{IJKO} f^O{}_{LMN} Z^I_A Z^J_B Z^{[K}_{[C} Z^{L]}_{D]} Z^M_E Z^N_F. \qquad \text{(B.6)}$$

On the other hand, substituting the first line of (3.22) ($-\varepsilon^{ABCD} = \gamma^{AC}\gamma^{BD} - \gamma^{BC}\gamma^{AD} + \gamma^{BA}\gamma^{CD}$) into the RHS of (B.6), we obtain

$$-I = 4P_1 - 2P_2 + P_3. \qquad \text{(B.7)}$$

Combining (B.4), (B.5) and (B.7), we find that

$$P_1 - P_2 = -\frac{2}{5} I. \qquad \text{(B.8)}$$

Substituting the above equation into Eq. (B.2), we reach the desired result:

$$-V' = -\frac{4}{45} I. \qquad \text{(B.9)}$$

Recall that we denote the last three terms of (B.1) as $-V'$, so the bosonic potential (B.1) is indeed $Sp(4)$ invariant. After some work, we reach the final expression for the bosonic potential (B.1):

$$-V = \frac{1}{60}(2f_{IJK}{}^O f_{OLMN} - 9f_{KLI}{}^O f_{ONMJ} + 2f_{IJL}{}^O f_{OKMN}) Z^N_A Z^{AI} Z^J_B Z^{BK} Z^L_C Z^{CM}. \qquad \text{(B.10)}$$

# APPENDIX C

# SOME EXPLICIT EXAMPLES OF $\mathcal{N}=5,6$ THEORIES

## C.1 $\mathcal{N}=5, Sp(2N)\times O(M)$ CSM theory

To generate a direct product gauge group, such as $Sp(2N)\times O(M)$, we first split one 3-algebra index into two indices: $I\rightarrow k\hat{k}$. As a result, a 3-algebra valued field becomes $Z^I_A\rightarrow Z^{k\hat{k}}_A$. We also decompose the antisymmetric tensor as $\omega_{IJ}\rightarrow\omega_{\hat{k}\hat{l}}\delta_{kl}$, where $\omega_{\hat{k}\hat{l}}$ is antisymmetric, and require $Z^{k\hat{k}}_A$ to be valued in the bifundamental representation of $Sp(2N)\times O(M)$. (Here $k,l=1,\cdots,M$ are the $O(M)$ indices while $\hat{k},\hat{l}=1,\cdots,2N$ the $Sp(2N)$ indices.) With this decomposition of $\omega_{IJ}$, we can rewrite the reality condition as

$$Z^{\dagger A}_{\hat{k}k}\equiv\omega^{AB}\omega_{\hat{k}\hat{l}}\delta_{kl}Z^{l\hat{l}}_B, \tag{C.1}$$

and similar conditions for the fermion and gauge fields. Consequently, the hermitian bilinear form of two fields

$$\omega^{AB}\omega_{IJ}Z^J_BZ^I_A=Z^{*I}_AZ^I_A=\bar{Z}^A_IZ^I_A, \tag{C.2}$$

can be rewritten in a trace form:

$$Z^{\dagger A}_{\hat{k}k}Z^{k\hat{k}}_A=\mathrm{Tr}(Z^{\dagger A}Z_A) \tag{C.3}$$

We then specify the 3-brackets as follows:

$$[T_{k\hat{k}},T_{l\hat{l}};T_{m\hat{m}}]=k(\delta_{kl}\omega_{\hat{k}\hat{m}}T_{m\hat{l}}+\delta_{kl}\omega_{\hat{l}\hat{m}}T_{m\hat{k}}-\delta_{km}\omega_{\hat{k}\hat{l}}T_{l\hat{m}}+\delta_{lm}\omega_{\hat{k}\hat{l}}T_{k\hat{m}}). \tag{C.4}$$

Of course, if we realize the generators of the 3-algebra $T_{k\hat{k}}$ as the fermionic generators of the super Lie algebra $OSp(M|2N)$, i.e., $T_{k\hat{k}}\doteq Q_{k\hat{k}}$, then the 3-bracket is realized in terms of the double graded bracket:

$$\begin{aligned}[T_{k\hat{k}},T_{l\hat{l}};T_{m\hat{m}}] &\doteq [\{Q_{k\hat{k}},Q_{l\hat{l}}\},Q_{m\hat{m}}] \\ &= k(\delta_{kl}\omega_{\hat{k}\hat{m}}Q_{m\hat{l}}+\delta_{kl}\omega_{\hat{l}\hat{m}}Q_{m\hat{k}}-\delta_{km}\omega_{\hat{k}\hat{l}}Q_{l\hat{m}}+\delta_{lm}\omega_{\hat{k}\hat{l}}Q_{k\hat{m}}).\end{aligned} \tag{C.5}$$

The overall coefficient $k$ on the right-hand side of Eq. (C.4) is assumed to be a real constant. It is straightforward to verify that the 3-brackets satisfy the FI (2.6) and the constraints (2.17). The corresponding structure constants are

$$f_{\hat{k}k,\hat{l}l,\hat{m}m,\hat{n}n} = k[(\delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm})\omega_{\hat{k}\hat{l}}\omega_{\hat{m}\hat{n}} - \delta_{kl}\delta_{mn}(\omega_{\hat{k}\hat{m}}\omega_{\hat{l}\hat{n}} + \omega_{\hat{k}\hat{n}}\omega_{\hat{l}\hat{m}})]. \tag{C.6}$$

It is not hard to check that the structure constants have the symmetry properties (2.18), and satisfy the reality condition (2.15). With this choice of structure constants, the gauge fields become: (We re-scale $A^{IJ}_\mu$ by $\frac{1}{k}$.)

$$\begin{aligned}
\tilde{A}_\mu{}^{m\hat{m}}{}_{n\hat{n}} &= A_\mu{}^{k\hat{k},l\hat{l}} f_{\hat{k}k,\hat{l}l}{}^{m\hat{m}}{}_{\hat{n}n} \\
&= -(A_{\mu\hat{n}\hat{l}}{}^{l\hat{m}} + A_\mu{}^{l\hat{m}}{}_{\hat{n}l})\delta^m{}_n + (A_{\mu\hat{l}n}{}^{m\hat{l}} + A_\mu{}^{m\hat{l}}{}_{\hat{l}n})\delta^{\hat{m}}{}_{\hat{n}} \\
&\equiv -(A_{\mu\hat{n}}{}^{\hat{m}} + A_\mu{}^{\hat{m}}{}_{\hat{n}})\delta^m{}_n + (-B_{\mu n}{}^m + B_\mu{}^m{}_n)\delta^{\hat{m}}{}_{\hat{n}} \\
&\equiv \hat{A}_\mu{}^{\hat{m}}{}_{\hat{n}}\delta^m{}_n + A_\mu{}^m{}_n\delta^{\hat{m}}{}_{\hat{n}}.
\end{aligned} \tag{C.7}$$

It is easy to see that $\hat{A}_\mu{}^{\hat{m}}{}_{\hat{n}}$ is the $Sp(2N)$ part of the gauge potential, because it can be written as $A^{\hat{k}\hat{l}}_\mu(t_{\hat{k}\hat{l}})^{\hat{m}}{}_{\hat{n}}$, where $(t_{\hat{k}\hat{l}})^{\hat{m}}{}_{\hat{n}}$ is the fundamental representation of the ordinary Lie algebra $Sp(2N)$. Similarly, we can identify $A_\mu{}^m{}_n$ as the $O(M)$ part of the gauge potential. As we explained in [39], the Lie algebra of the gauge group $Sp(2N)\times O(M)$ is actually generated by the FI (2.6) after we specify the structure constants by Eq. (C.6).

We would like to derive the $\mathcal{N}=5, Sp(2N)\times O(M)$ Lagrangian and the corresponding supersymmetry transformation law in the 3-algebraic framework. With the notation (C.3), the kinetic terms for matter fields in the Lagrangian (3.38) read

$$-\frac{1}{2}\mathrm{Tr}(D_\mu Z^{\dagger A}D^\mu Z_A - i\bar{\psi}^{\dagger A}D_\mu\gamma^\mu\psi_A). \tag{C.8}$$

With the choice of the structure constants (C.6), we learn that

$$f_{IJKL}X^IY^JZ^KW^L = -k\mathrm{Tr}(\mathrm{XY^\dagger ZW^\dagger + YX^\dagger ZW^\dagger - ZX^\dagger YW^\dagger - ZY^\dagger XW^\dagger}). \tag{C.9}$$

Hence the Yukawa terms in the Lagrangian (3.38) become

$$\begin{aligned}
&ik\varepsilon^{ABCD}\mathrm{Tr}(Z_A\bar{\psi}^\dagger_B Z_C\psi^\dagger_D) \\
&-i\frac{k}{2}\mathrm{Tr}(\bar{\psi}^\dagger_A Z_B Z^{\dagger B}\psi^A - \bar{\psi}_A Z^\dagger_B Z^B\psi^{\dagger A} - 2\bar{\psi}^\dagger_A Z_B Z^{\dagger A}\psi^B + 2\bar{\psi}^A Z^{\dagger B}Z_A\psi^\dagger_B),
\end{aligned} \tag{C.10}$$

where we have used the following $Sp(4)$ identity:

$$\varepsilon^{ABCD} = -\omega^{AB}\omega^{CD} + \omega^{AC}\omega^{BD} - \omega^{AD}\omega^{BC}. \tag{C.11}$$

The Chern-Simons term in the Lagrangian (3.38) can be written as

$$\frac{1}{2}\epsilon^{\mu\nu\lambda}(\tilde{A}_{\mu}^{IJ}\partial_{\nu}A_{\lambda IJ}+\frac{2}{3}\tilde{A}_{\mu}{}^{I}{}_{J}\tilde{A}_{\nu}{}^{J}{}_{L}A_{\lambda}{}^{L}{}_{I}). \tag{C.12}$$

Substituting the definition of the gauge fields (C.7) into the above equation gives the conventional Chern-Simons term

$$\frac{1}{4k}\epsilon^{\mu\nu\lambda}\mathrm{Tr}(\hat{A}_{\mu}\partial_{\nu}\hat{A}_{\lambda}+\frac{2}{3}\hat{A}_{\mu}\hat{A}_{\nu}\hat{A}_{\lambda}-A_{\mu}\partial_{\nu}A_{\lambda}-\frac{2}{3}A_{\mu}A_{\nu}A_{\lambda}). \tag{C.13}$$

Finally we want to calculate the potential terms in the Lagrangian (3.38). By using $f_{IJKL}=f_{IJLK}$, they can be rewritten as

$$\begin{aligned}&\frac{1}{60}\mathrm{Tr}(2[Z^{A},Z_{B};Z^{B}][Z^{C},Z_{A};Z_{C}]^{\dagger}-9[Z^{B},Z_{C};Z^{A}][Z^{C},Z_{B};Z_{A}]^{\dagger}\\&\qquad+2[Z^{A},Z_{B};Z_{C}][Z^{C},Z_{A};Z^{B}]^{\dagger}).\end{aligned} \tag{C.14}$$

The last two terms can be combined together:

$$\begin{aligned}&-\frac{4k^{2}}{15}(\omega_{AF}\omega_{BE}\omega_{CD}-2\omega_{AF}\omega_{BC}\omega_{DE}+2\omega_{AC}\omega_{BF}\omega_{DE}-\omega_{AE}\omega_{BF}\omega_{CD}\\&\qquad+\omega_{AC}\omega_{BE}\omega_{DF}-\omega_{AE}\omega_{BC}\omega_{DF})\mathrm{Tr}(Z^{B}Z^{\dagger D}Z^{A}Z^{\dagger C}Z^{E}Z^{\dagger F}).\end{aligned} \tag{C.15}$$

The first term becomes

$$\begin{aligned}&\frac{k^{2}}{30}(2\omega_{AD}\omega_{BE}\omega_{CF}+4\omega_{AB}\omega_{CF}\omega_{DE}-2\omega_{AE}\omega_{BD}\omega_{CF}+\omega_{AD}\omega_{BC}\omega_{EF}\\&\qquad-2\omega_{AB}\omega_{CD}\omega_{EF}-\omega_{AC}\omega_{BD}\omega_{EF}+\omega_{AD}\omega_{BF}\omega_{CE}+2\omega_{AB}\omega_{CE}\omega_{DF}\\&\qquad-\omega_{AF}\omega_{BD}\omega_{CE})\mathrm{Tr}(Z^{B}Z^{\dagger D}Z^{A}Z^{\dagger C}Z^{E}Z^{\dagger F}).\end{aligned} \tag{C.16}$$

Clearly, they can be simplified further. Taking account of the cyclic property of the trace, there are only four possible potential terms:

$$\begin{aligned}&(c_{1}\omega_{AD}\omega_{BE}\omega_{CF}+c_{2}\omega_{BD}\omega_{CE}\omega_{AF}+c_{3}\omega_{AD}\omega_{CE}\omega_{BF}+c_{4}\omega_{CD}\omega_{AE}\omega_{BF})\\&\quad\times\mathrm{Tr}(Z^{A}Z^{\dagger D}Z^{B}Z^{\dagger E}Z^{C}Z^{\dagger F}),\end{aligned} \tag{C.17}$$

where $c_1,\cdots$ and $c_4$ are constants. After some work, we reach the final expression for the potential:

$$\begin{aligned}&\frac{k^{2}}{6}\mathrm{Tr}(-6Z_{A}Z^{\dagger A}Z_{B}Z^{\dagger C}Z_{C}Z^{\dagger B}+4Z_{A}Z^{\dagger C}Z_{B}Z^{\dagger A}Z_{C}Z^{\dagger B}\\&\qquad+Z_{A}Z^{\dagger A}Z_{B}Z^{\dagger B}Z_{C}Z^{\dagger C}+Z_{A}Z^{\dagger B}Z_{B}Z^{\dagger C}Z_{C}Z^{\dagger A}).\end{aligned} \tag{C.18}$$

In deriving this potential, we have used another $Sp(4)$ identity [35]:

$$\begin{aligned}
\varepsilon_{GABC}\varepsilon^{GDEF} &= 3!\delta^D_{[A}\delta^E_B\delta^F_{C]} \\
&= 3(-\delta^D_{[A}\omega^{EF}\omega_{BC]} + \delta^E_{[A}\omega^{DF}\omega_{BC]} - \delta^F_{[A}\omega^{DE}\omega_{BC]}).
\end{aligned} \tag{C.19}$$

In summary, with the choice of the structure constants (C.6), the Lagrangian (3.38) is given by

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{2}\mathrm{Tr}(D_\mu Z^{\dagger A} D^\mu Z_A - i\bar{\psi}^{\dagger A} D_\mu \gamma^\mu \psi_A) + ik\varepsilon^{ABCD}\mathrm{Tr}(Z_A \bar{\psi}^\dagger_B Z_C \psi^\dagger_D) \\
& -i\frac{k}{2}\mathrm{Tr}(\bar{\psi}^\dagger_A Z_B Z^{\dagger B}\psi^A - \bar{\psi}_A Z^\dagger_B Z^B \psi^{\dagger A} - 2\bar{\psi}^\dagger_A Z_B Z^{\dagger A}\psi^B + 2\bar{\psi}^A Z^{\dagger B} Z_A \psi^\dagger_B) \\
& +\frac{1}{4k}\epsilon^{\mu\nu\lambda}\mathrm{Tr}(\hat{A}_\mu \partial_\nu \hat{A}_\lambda + \frac{2}{3}\hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda - A_\mu \partial_\nu A_\lambda - \frac{2}{3}A_\mu A_\nu A_\lambda) \\
& +\frac{k^2}{6}\mathrm{Tr}(-6Z_A Z^{\dagger A} Z_B Z^{\dagger C} Z_C Z^{\dagger B} + 4 Z_A Z^{\dagger C} Z_B Z^{\dagger A} Z_C Z^{\dagger B} \\
& \qquad + Z_A Z^{\dagger A} Z_B Z^{\dagger B} Z_C Z^{\dagger C} + Z_A Z^{\dagger B} Z_B Z^{\dagger C} Z_C Z^{\dagger A}),
\end{aligned} \tag{C.20}$$

Substituting the structure constants ($C$.6) into (3.45), the SUSY transformation law reads

$$\begin{aligned}
\delta Z_A &= i\bar{\epsilon}_A{}^B \psi_B \\
\delta \psi_A &= \gamma^\mu D_\mu Z_B \epsilon^B{}_A - \frac{2k}{3}\epsilon^C{}_A (Z_{[B} Z^{\dagger B} Z_{C]} + Z_B Z^\dagger_C Z^B) \\
&\quad + \frac{4k}{3}\epsilon^C{}_B (Z_{[C} Z^{\dagger B} Z_{A]} + Z_C Z^\dagger_A Z^B) \\
\delta A_\mu &= ik\bar{\epsilon}^{AB}\gamma_\mu (Z_A \psi^\dagger_B + \psi_B Z^\dagger_A) \\
\delta \hat{A}_\mu &= -ik\bar{\epsilon}^{AB}\gamma_\mu (\psi^\dagger_B Z_A + Z^\dagger_A \psi_B).
\end{aligned} \tag{C.21}$$

The $\mathcal{N}=5, Sp(2N)\times O(M)$ Lagrangian (C.20) and the supersymmetry transformation law (C.21) are in agreement with those given in Ref. [35], which were derived in terms of ordinary Lie algebra. This theory has been conjectured to be the dual gauge theory of M2 branes probing a $\mathbf{C}^4/\hat{\mathbf{D}}_k$ singularity, where $\hat{\mathbf{D}}_k$ is the binary dihedral group [35, 36].

## C.2 SUSY Transformation Law and Lagrangian in $D=3$, $\mathcal{N}=6$ CSM Theories

For this thesis to be self contained, below we give the explicit form of the SUSY transformation law and the Lagrangian for the $D=3, \mathcal{N}=6$ CSM theories with $SU(4)$ $R$-symmetry. For the notations, see subsections 7.4.1 and 7.4.2.

### C.2.1 $Sp(2N) \times U(1)$ CSM Theory

The Lagrangian of the theory is given by

$$\begin{aligned}
\mathcal{L} &= -D_\mu \bar{Z}^a_A D^\mu Z^A_a - i\bar{\psi}^{Aa}\gamma^\mu D_\mu \psi_{Aa} \\
&+ik(\bar{Z}^b_B \bar{\psi}_{Ab}\psi^{Aa}Z^B_a - \bar{Z}^b_B Z^B_b \bar{\psi}^{Aa}\psi_{Aa} - \bar{Z}^c_B \omega_{cd}\bar{\psi}^{Ad}\psi_{Aa}\omega^{ab}Z^B_b) \\
&-2ik(\bar{Z}^b_B \bar{\psi}_{Ab}\psi^{Ba}Z^A_a - \bar{Z}^b_B Z^A_b \bar{\psi}^{Ba}\psi_{Aa} - \bar{Z}^c_B \omega_{cd}\bar{\psi}^{Bd}\psi_{Aa}\omega^{ab}Z^A_b) \\
&-ik\varepsilon^{ABCD}(\bar{Z}^a_A \bar{\psi}_{Ba}\bar{Z}^b_C \psi_{Db} - \frac{1}{2}\bar{Z}^c_A \omega_{cd}\bar{Z}^d_C \bar{\psi}_{Ba}\omega^{ab}\psi_{Db}) \\
&-ik\varepsilon_{ABCD}(\bar{\psi}^{Ba}Z^A_a \psi^{Db}Z^C_b - \frac{1}{2}Z^A_a \omega^{ab}Z^C_b \bar{\psi}^{Bc}\omega_{cd}\psi^{Dd}) \\
&+\frac{1}{2k}\varepsilon^{\mu\nu\lambda}A_\mu \partial_\nu A_\lambda - \frac{1}{4k}\varepsilon^{\mu\nu\lambda}\mathrm{Tr}(B_\mu \partial_\nu B_\lambda + \frac{2}{3}B_\mu B_\nu B_\lambda) \\
&-3k^2 Z^B_a \omega^{ab}Z^D_b \bar{Z}^e_D Z^A_e \bar{Z}^c_A \omega_{cd}\bar{Z}^d_B + \frac{5k^2}{3}\bar{Z}^a_A Z^B_a \bar{Z}^b_B Z^D_b \bar{Z}^c_D Z^A_c \\
&-2k^2 \bar{Z}^a_A Z^B_a \bar{Z}^b_D Z^D_b \bar{Z}^c_B Z^A_c + \frac{k^2}{3}\bar{Z}^a_B Z^B_a \bar{Z}^b_D Z^D_b \bar{Z}^c_A Z^A_c .
\end{aligned} \tag{C.22}$$

The SUSY transformation laws are given by

$$\begin{aligned}
\delta Z^A_d &= -i\bar{\epsilon}^{AB}\psi_{Bd} \\
\delta \bar{Z}^d_A &= -i\bar{\epsilon}_{AB}\psi^{Bd} \\
\delta \psi_{Bd} &= \gamma^\mu D_\mu Z^A_d \epsilon_{AB} - kZ^C_a \omega^{ab}Z^A_b \omega_{dc}\bar{Z}^c_C \epsilon_{AB} - kZ^C_a \omega^{ab}Z^D_b \omega_{dc}\bar{Z}^c_B \epsilon_{CD} \\
&\quad -k\bar{Z}^a_C Z^C_a Z^A_d \epsilon_{AB} + k\bar{Z}^a_C Z^A_a Z^C_d \epsilon_{AB} - 2k\bar{Z}^a_B Z^C_a Z^D_d \epsilon_{CD} \\
\delta \psi^{Bd} &= \gamma^\mu D_\mu \bar{Z}^d_A \epsilon^{AB} - k\bar{Z}^a_C \omega_{ab}\bar{Z}^b_A \omega^{dc}Z^C_c \epsilon^{AB} - k\bar{Z}^a_C \omega_{ab}\bar{Z}^b_D \omega^{dc}Z^B_c \epsilon^{CD} \\
&\quad -k\bar{Z}^a_C Z^C_a \bar{Z}^d_A \epsilon^{AB} + k\bar{Z}^a_A Z^C_a \bar{Z}^d_C \epsilon^{AB} - 2k\bar{Z}^a_C Z^B_a \bar{Z}^d_D \epsilon^{CD} \\
\delta A_\mu &= -ik\bar{\epsilon}_{AB}\gamma_\mu \psi^{Ba}Z^A_a + ik\bar{\epsilon}^{AB}\gamma_\mu \bar{Z}^a_A \psi_{Ba} \\
\delta {B_\mu}^c{}_d &= ik\bar{\epsilon}_{AB}\gamma_\mu \omega^{ca}Z^A_a \omega_{db}\psi^{Bb} - ik\bar{\epsilon}^{AB}\gamma_\mu \omega_{db}\bar{Z}^b_A \omega^{ca}\psi_{Ba} \\
&\quad +ik\bar{\epsilon}_{AB}\gamma_\mu Z^A_d \psi^{Bc} - ik\bar{\epsilon}^{AB}\gamma_\mu \bar{Z}^c_A \psi_{Bd}.
\end{aligned} \tag{C.23}$$

### C.2.2 $U(M) \times U(N)$ CSM Theory

The Lagrangian of the theory is given by

$$\begin{aligned}
\mathcal{L} &= -\mathrm{Tr}(D_\mu \bar{Z}_A D^\mu Z^A) - i\mathrm{Tr}(\bar{\psi}^A \gamma^\mu D_\mu \psi_A) - V + \mathcal{L}_{CS} \\
&-ik\mathrm{Tr}(\bar{\psi}^A \psi_A \bar{Z}_B Z^B - \bar{\psi}^A Z^B \bar{Z}_B \psi_A) \\
&+2ik\mathrm{Tr}(\bar{\psi}^A \psi_B \bar{Z}_A Z^B - \bar{\psi}^A Z^B \bar{Z}_A \psi_B) \\
&+ik\varepsilon_{ABCD}\mathrm{Tr}(\bar{\psi}^A Z^C \bar{\psi}^B Z^D) - ik\varepsilon^{ABCD}\mathrm{Tr}(\bar{Z}_D \psi_A \bar{Z}_C \psi_B) .
\end{aligned} \tag{C.24}$$

The Lagrangian (C.24) is the same obtained by BL [38], except for that we re-scale the gauge fields by a factor $\frac{1}{k}$. The potential term is

$$\begin{aligned} V &= 2k^2\mathrm{Tr}(\bar{Z}_A Z^A \bar{Z}_B Z^C \bar{Z}_C Z^B) - \frac{4k^2}{3}\mathrm{Tr}(Z^A \bar{Z}_B Z^C \bar{Z}_A Z^B \bar{Z}_C) \\ & -\frac{k^2}{3}\mathrm{Tr}(Z^A \bar{Z}_A Z^B \bar{Z}_B Z^C \bar{Z}_C + \bar{Z}_A Z^A \bar{Z}_B Z^B \bar{Z}_C Z^C). \end{aligned} \tag{C.25}$$

The Chern-Simons term reads

$$\mathcal{L}_{CS} = \frac{1}{2k}\varepsilon^{\mu\nu\lambda}\mathrm{Tr}\left(\hat{A}_\mu \partial_\nu \hat{A}_\lambda + \frac{2}{3}\hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda - A_\mu \partial_\nu A_\lambda - \frac{2}{3}A_\mu A_\nu A_\lambda\right). \tag{C.26}$$

The $\mathcal{N} = 6$ SUSY transformation laws, which are closed on-shell with the equations of motion derivable from the above lagrangian (C.24), are given by

$$\begin{aligned} \delta Z^A &= -i\bar{\epsilon}^{AB}\psi_B \\ \delta \bar{Z}_A &= -i\bar{\epsilon}_{AB}\bar{\psi}^B \\ \delta \psi_B &= \gamma^\mu D_\mu Z^A \epsilon_{AB} + k(Z^C \bar{Z}_C Z^A - Z^A \bar{Z}_C Z^C)\epsilon_{AB} + 2k Z^C \bar{Z}_B Z^D \epsilon_{CD} \\ \delta \bar{\psi}^B &= \gamma^\mu D_\mu \bar{Z}_A \epsilon^{AB} + k(\bar{Z}_A Z^C \bar{Z}_C - \bar{Z}_C Z^C \bar{Z}_A)\epsilon^{AB} + 2k\bar{Z}_D Z^B \bar{Z}_C \epsilon^{CD} \\ \delta \hat{A}_\mu &= -ik\bar{\epsilon}_{AB}\gamma_\mu \bar{\psi}^B Z^A + ik\bar{\epsilon}^{AB}\gamma_\mu \bar{Z}_A \psi_B \\ \delta A_\mu &= ik\bar{\epsilon}_{AB}\gamma_\mu Z^A \bar{\psi}^B - ik\bar{\epsilon}^{AB}\gamma_\mu \psi_B \bar{Z}_A. \end{aligned} \tag{C.27}$$

## REFERENCES


[1] B. M. Zupnik and D. G. Pak, "Superfield formulation of the simpliest three-dimensional gauge theories and conformal supergravities, " Theor. Math. Phys. **77** 1070 (1988) [Teor. Mat. Fit. **77**, 97 (1988 )]

[2] E. A. Ivanov, "Chern-Simons matter systems with manifest $\mathcal{N}$=2 supersymmetry," Phys. Lett. B **268**, 203 (1991).

[3] L. V. Avdeev, D. I. Kazakov and I. N. Kondrashuk, "Renormalizations in supersymmetric and nonsupersymmetric nonAbelian Chern-Simons field theories with matter," Nucl. Phys. B **391**, 333 (1993).

[4] L. V. Avdeev, G. V. Grigorev and D. I. Kazakov, "Renormalizations in Abelian Chern-Simons field theories with matter," Nucl. Phys. B **382**, 561 (1992).

[5] H. C. Kao, K. M. Lee and T. Lee, "The Chern-Simons coefficient in supersymmetric Yang-Mills Chern-Simons theories," Phys. Lett. B **373**, 94 (1996) [arXiv:hep-th/9506170].

[6] A. Kapustin and M. J. Strassler, "On mirror symmetry in three dimensional Abelian gauge theories," J. High Energy Physics **9904**, 021 (1999) [arXiv:hep-th/9902033].

[7] W. Chen, G.W. Semenoff and Y.S. Wu, "Scale and conformal invariance in Chern-Simons-matter field theory," Phys. Rev. D44, **1625** (1991).

[8] W. Chen, G. Semenoff and Y.S. Wu, "Probing Topological Features in Perturbative Chern-Simons Gauge Theory," Mod. Phys. Lett. A5, **1833** (1990).

[9] W. Chen, G.W. Semenoff and Y.S. Wu, "Two-Loop Analysis of Chern-Simons-matter Theory," Phys. Rev. D46; **5521** (1992), arXiv:hep-th/9209005.

[10] O.M. Del Cima, D.H.T. Franco, J.A. Helayel-Neto and O. Piguet, "An algebraic proof on the finiteness of Yang-Mills-Chern-Simons theory in D=3," Lett. Math. Phys. **47**, 265 (1999); arXiv:math-ph/9904030.

[11] Nikolas Akerblom, Christian Saemann and Martin Wolf, "Marginal Deformations and 3-Algebra Structures," arXiv:0906.1705 [hep-th].

[12] Y. Nambu, Generalized Hamiltonian mechanics, Phys. Rev. **D7** (1973), 2405-2412.

[13] H. Awata, M. Li, D. Minic, and T. Yoneya, "On the quantization of nambu brackets," arXiv:hep-th/9906248.

[14] J. Bagger and N. Lambert, "Modeling multiple M2's," Phys. Rev. D **75**, 045020 (2007), arXiv:hep-th/0611108; J. Bagger and N. Lambert, "Gauge symmetry and supersymmetry of multiple M2-branes," Phys. Rev. D **77**, 065008 (2008),

arXiv:0711.0955 [hep-th]; J. Bagger and N. Lambert, "Comments on multiple M2-branes," J. High Energy Physics **0802**, 105 (2008), arXiv:0712.3738 [hep-th].

[15] A. Gustavsson, "Algebraic structures on parallel M2-branes," arXiv:0709.1260 [hep-th]; "Selfdual strings and loop space Nahm equations," J. High Energy Physics **0804**, 083 (2008), arXiv:0802.3456 [hep-th].

[16] J. Distler, S. Mukhi, C. Papageorgakis and M. Van Raamsdonk, "M2-branes on M-folds," J. High Energy Physics **0805**, 038 (2008) arXiv:0804.1256 [hep-th].

[17] N. Lambert and D. Tong, "Membranes on an Orbifold," arXiv:0804.1114 [hep-th].

[18] Chethan Krishnan and Carlo Maccaferri, "Membranes on calibrations," J. High Energy Physics **0807**, 005 (2008) arXiv:0805.3125 [hep-th].

[19] Giulio Bonelli, Alessandro Tanzini and Maxim Zabzine, "Topological branes, p-algebras and generalized Nahm equations," arXiv:0807.5113 [hep-th].

[20] Jerome P. Gauntlett and Jan B. Gutowski, "Constraining maximally supersymmetric membrane actions," arXiv:0804.3078 [hep-th].

[21] G. Papadopoulos, "M2-branes, 3-Lie Algebras and Plucker relations ," arXiv:0804.2662 [hep-th].

[22] J. Gomis, G. Milanesi and J. G. Russo, "Bagger-Lambert theory for general Lie algebras," arXiv:0805.1012 [hep-th]; S. Benvenuti, D. Rodriguez-Gomez, E. Tonni and H. Verlinde, "$\mathcal{N}$=8 superconformal gauge theories and M2 branes," arXiv:0805.1087 [hep-th]; P. M. Ho, Y. Imamura and Y. Matsuo, "M2 to D2 revisited," arXiv:0805.1202 [hep-th].

[23] M. A. Bandres, A. E. Lipstein and J. H. Schwarz, "Ghost-free superconformal action for multiple M2-branes," arXiv:0806.0054 [hep-th].

[24] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk and H. Verlinde, "Supersymmetric Yang-Mills theory from Lorentzian three-algebras," arXiv:0806.0738 [hep-th].

[25] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, "$\mathcal{N}$=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," arXiv:0806.1218 [hep-th].

[26] Marcus K. Benna, Igor R. Klebanov, and Thomas Klose, "Charges of monopole operators in Chern-Simons Yang-Mills theory," arXiv:0906.3008 [hep-th].

[27] Andreas Gustavsson and Soo-Jong Rey, "Enhanced $\mathcal{N}$=8 supersymmetry of ABJM theory on R(8) and R(8)/Z(2)," arXiv:0906.3568 [hep-th].

[28] O-Kab Kwon, Phillial Oh and Jongsu Sohn, "Notes on supersymmetry enhancement of ABJM theory," arXiv:0906.4333 [hep-th].

[29] M. Benna, I. Klebanov, T. Klose and M. Smedback, "Superconformal Chern-Simons theories and $AdS_4/CFT_3$ correspondence," arXiv:0806.1519 [hep-th].

[30] M. Bandres, A. Lipstein and J. Schwarz "Studies of the ABJM theory in formulation with manifest $SU(4)$ R-symmetry," arXiv:0807.0880 [hep-th].

[31] E. A. Bergshoeff, M. de Roo and O. Hohm, "Multiple M2-branes and the embedding tensor," arXiv:0804.2201 [hep-th]; E. A. Bergshoeff, M. de Roo, O. Hohm and D. Roest, "Multiple membranes from gauged supergravity," arXiv:0806.2584 [hep-th].

[32] E. A. Bergshoeff, O. Hohm, D. Roest, H. Samtleben and E. Sezgin, "The superconformal gaugings in three dimensions," arXiv:0807.2841 [hep-th].

[33] D. Gaiotto and E. Witten, "Janus configurations, Chern-Simons couplings, and the theta-angle in $\mathcal{N}$=4 super Yang-Mills theory," arXiv:0804.2907 [hep-th].

[34] K. Hosomichi, K. M. Lee, S. Lee, S. Lee and J. Park, "$\mathcal{N}$=4 superconformal Chern-Simons theories with hyper and twisted hyper multiplets," arXiv:0805.3662 [hep-th].

[35] K. Hosomichi, K. M. Lee, S. Lee, S. Lee and J. Park, "$\mathcal{N}$=5,6 superconformal Chern-Simons theories and M2-branes on orbifolds," arXiv:0806.4977 [hep-th].

[36] O. Aharony, O. Bergman and D. L. Jafferis, "Fractional M2-branes," arXiv:0807.4924 [hep-th].

[37] Paul de Medeiros, José Figueroa-O'Farrill and Elena Méndez-Escobar, "Superpotentials for superconformal Chern-Simons theories from representation theory," arXiv:0908.2125 [hep-th].

[38] J. Bagger and N. Lambert, "Three-algebras and $\mathcal{N}$=6 Chern-Simons gauge theories," arXiv:0807.0163 [hep-th].

[39] Fa-Min Chen and Yong-Shi Wu, "Symplectic three-algebra and $\mathcal{N} = 6, Sp(2N) \times U(1)$ superconformal Chern-Simons-matter theory ," version accepted for publication in European Physics Journal C, arXiv:0902.3454 [hep-th].

[40] Fa-Min Chen, "Symplectic three-algebra unifying $\mathcal{N} = 5, 6$ superconformal Chern-Simons-matter theories," J. High Energy Physics 1008:077,2010, arXiv:0908.2618 [hep-th]

[41] S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel, " Superspace, or one thousand and one lessons in supersymmetry," Front. Phys. **58** (1983) 1-548, hep-th/0108200.

[42] José Figueroa-O'Farrill, "Simplicity in the Faulkner construction," arXiv:0905.4900 [hep-th].

[43] Paul de Medeiros, José Figueroa-O'Farrill, Elena Méndez-Escobar and Patricia Ritter, "On the Lie-algebraic origin of metric 3-algebras," arXiv:0809.1086 [hep-th].

[44] Fa-Min Chen and Yong-Shi Wu, "Superspace formulation in a three-algebra approach to $D = 3, \mathcal{N} = 4, 5$ superconformal Chern-Simons-matter theories," arXiv: 1007.5157 [hep-th].

[45] Fa-Min Chen and Yong-Shi Wu, " $D = 3, \mathcal{N} = 4$ quiver gauge theory in terms of the bosonic part of a superalgebra 'fused' by two simple superalgebras," in preparation.

[46] Jakob Palmkvist, "Three-algebras, triple systems and 3-graded Lie superalgebras," arXiv:0905.2468 [hep-th].

[47] J. Bagger and G. Bruhn, "Three-algebras in $\mathcal{N} = 5, 6$ superconformal Chern-Simons theories: Representations and relations," arXiv:1006.0040 [hep-th].

[48] M. Schnabl and Y. Tachikawa, "Classifiction of $\mathcal{N} = 6$ superconformal theories of ABJM type," arXiv:0807.1102 [hep-th].

[49] V. G. Kac, "Lie superalgebras," Adv. Math. **26** (1977) 8.

[50] S. Cherkis and C. Saemann, "Multiple M2-branes and generalized 3-Lie algebras," Phys. Rev. D 78: 066019 (2008), arXiv:0807.0808 [hep-th].